A COMPANION

TO

WRIGLEY'S COLLECTION

OF

EXAMPLES AND PROBLEMS

BEING ILLUSTRATIONS OF MATHEMATICAL PROCESSES AND METHODS OF SOLUTION.

BY

(Thompson)

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CAMBRIDGE:

DEIGHTON, BELL, AND CO.

LONDON: BELL AND DALDY.

1861.

Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

PREFACE.

THE object of this work is, not to furnish to Wrigley's Collection a Key in which every Example and Problem is worked out and solved, but to exemplify Mathematical Operations and Methods of Solution by numerous illustrations. A Key, it is judged inexpedient to publish, on the ground that it would destroy or impair the value of the Collection as an Exercise book. But a work, in which principles are illustrated and processes set forth, such as would serve to solve every question in the Collection, will, it is anticipated, prove convenient for the Tutor and especially useful for the Student.

The Arithmetic and the beginning of the Algebra in the Collection are left untouched.

Any list of Errata, sent to the Publishers, would be gratefully received.

September 21, 1861.

CONTENTS.

The following are the numbers of the Examples and Problems in the Fourth and subsequent Editions of the Collection discussed in this work.

ALGEBRA.

THEORY OF EQUATIONS.

GEOMETRY.

MENSURATION.

PLANE TRIGONOMETRY.

APPLICATION OF ALGEBRA TO GEOMETRY.

ANALYTICAL GEOMETRY AND CONIC SECTIONS.

STATICS.

DYNAMICS.

HYDROSTATICS.

HYDRODYNAMICS.

ERRATA.

PAGE

89 (14) Change the sign in X_2 thus, $+7x^2-8x-4$; hence, the roots are real, and lie as follows $\{-2, -1\}$; $\{-1, 0\}$; $\{1, 2\}$; $\{1, 2\}$.

90 (19) The signs in X_5, X_6, X_7 should all be changed; hence there will be 2 pairs of imaginary roots, and the 3 real roots will be found to lie thus $\{-2, -1\}$; $\{1, 2\}$; $\{1, 2\}$.

91 (20) See the Appendix.

128 Insert *Ex.* 12 before (5).

184 Last line, for d read d'.

185 Fifth line, for πdd read $\pi dd'$.

270 Insert 19. at the beginning of the line—Let the axis, &c.

297 The problem (7) belongs to the Wheel and Axle, and should precede *Ex.* 12.

347 In probs. 25 and 27 for tan a read tan a.

ALGEBRA.

I. Simple Equations.

Ex. 35.

6. $$\frac{ax-b^2}{(ax)^{\frac{1}{2}}+b}-\frac{(ax)^{\frac{1}{2}}-b}{c}=c,\ \text{find } x;$$

$$\therefore\ (ax)^{\frac{1}{2}}-b-\frac{(ax)^{\frac{1}{2}}-b}{c}=c,$$

$$\{(ax)^{\frac{1}{2}}-b\}\left(1-\frac{1}{c}\right)=c;$$

$$\therefore\ (ax)^{\frac{1}{2}}-b=\frac{c^2}{c-1},$$

$$(ax)^{\frac{1}{2}}=b+\frac{c^2}{c-1};$$

$$x=\frac{1}{a}\left(b+\frac{c^2}{c-1}\right)^2.$$

8. $$\frac{1}{a}(a+x)^{\frac{1}{2}}+\frac{1}{x}(a+x)^{\frac{1}{2}}=\frac{1}{b}\,.\,x^{\frac{1}{2}},\ \text{find } x;$$

$$\therefore\ \frac{(a+x)(a+x)^{\frac{1}{2}}}{ax}=\frac{x^{\frac{1}{2}}}{b},$$

$$\text{or }\ (a+x)^{\frac{3}{2}}=\frac{ax^{\frac{3}{2}}}{b};$$

$$\therefore\ a+x=\frac{a^{\frac{2}{3}}x}{b^{\frac{2}{3}}};$$

$$\therefore\ x(a^{\frac{2}{3}}-b^{\frac{2}{3}})=ab^{\frac{2}{3}};$$

$$\therefore\ x=\frac{ab^{\frac{2}{3}}}{a^{\frac{2}{3}}-b^{\frac{2}{3}}}.$$

21. $$ax + 1 = \frac{2ax\,(x+a^2)^{\frac{1}{2}}}{a + (x+a^2)^{\frac{1}{2}}}, \text{ find } x.$$

$$\text{Then } \frac{2ax}{ax+1} = \frac{a}{(x+a^2)^{\frac{1}{2}}} + 1.$$

Transpose 1 and square,

$$\left(\frac{ax-1}{ax+1}\right)^2 = \frac{a^2}{x+a^2},$$

$$\left(\frac{ax+1}{ax-1}\right)^2 = \frac{x}{a^2} + 1,$$

$$\frac{4ax}{(ax-1)^2} = \frac{x}{a^2},$$

$$ax - 1 = \pm\sqrt{4a^3} = \pm\, 2a\sqrt{a};$$

$$\therefore\ x = \pm\, 2a^{\frac{1}{2}} + a^{-1}.$$

22. $$\frac{a^{\frac{1}{2}} - \{a - (a^2 - ax)^{\frac{1}{2}}\}^{\frac{1}{2}}}{a^{\frac{1}{2}} + \{a - (a^2 - ax)^{\frac{1}{2}}\}^{\frac{1}{2}}} = b, \text{ find } x.$$

Dividendo and Componendo,

$$\frac{\{a - (a - a^2x)^{\frac{1}{2}}\}^{\frac{1}{2}}}{a^{\frac{1}{2}}} = \frac{1-b}{1+b};$$

$$\therefore\ 1 - \frac{(a^2-ax)^{\frac{1}{2}}}{a} = \left(\frac{1-b}{1+b}\right)^2;$$

$$\therefore\ 1 - \frac{x}{a} = \left\{1 - \left(\frac{1-b}{1+b}\right)^2\right\}^2 = \frac{(4b)^2}{(1+b)^4};$$

$$\therefore\ x = a\left\{1 - \left(\frac{2b^{\frac{1}{2}}}{1+b}\right)^4\right\}.$$

24. $$\frac{1 + x - (2x+x^2)^{\frac{1}{2}}}{1 + x + (2x+x^2)^{\frac{1}{2}}} = a\,\frac{(2+x)^{\frac{1}{2}} + x^{\frac{1}{2}}}{(2+x)^{\frac{1}{2}} - x^{\frac{1}{2}}}, \text{ find } x;$$

$$\therefore\ \frac{\{(2+x)^{\frac{1}{2}} - x^{\frac{1}{2}}\}^2}{\{(2+x)^{\frac{1}{2}} + x^{\frac{1}{2}}\}^2} = a\,\frac{(2+x)^{\frac{1}{2}} + x^{\frac{1}{2}}}{(2+x)^{\frac{1}{2}} - x^{\frac{1}{2}}},$$

$$\frac{(2+x)^{\frac{1}{2}} - x^{\frac{1}{2}}}{(2+x)^{\frac{1}{2}} + x^{\frac{1}{2}}} = a^{\frac{1}{3}}.$$

Componendo and Dividendo,

$$\left(\frac{2+x}{x}\right)^{\frac{1}{2}} = \frac{1+a^{\frac{1}{3}}}{1-a^{\frac{1}{3}}};$$

$$\therefore \frac{2}{x} = \left(\frac{1+a^{\frac{1}{3}}}{1-a^{\frac{1}{3}}}\right)^2 - 1 = \frac{4a^{\frac{1}{3}}}{(1-a^{\frac{1}{3}})^2};$$

$$\therefore x = \frac{(1-a^{\frac{1}{3}})^2}{2a^{\frac{1}{3}}} = \tfrac{1}{2}\left(a^{\frac{1}{6}} - a^{-\frac{1}{6}}\right)^2.$$

II. Simultaneous Equations.

Ex. 36.

17. $$\left.\begin{aligned} \frac{1}{3x} + \frac{1}{5y} &= \frac{2}{9} \quad (1) \\ \frac{1}{5x} + \frac{1}{3y} &= \frac{1}{4} \quad (2) \end{aligned}\right\}, \text{ find } x \text{ and } y.$$

Multiply (1) by 5, (2) by 3, and subtract;

$$\therefore \left(\frac{5}{3} - \frac{3}{5}\right)\frac{1}{x} = \frac{10}{9} - \frac{3}{4};$$

$$x = \frac{25-9}{15} \div \frac{40-27}{36} = \frac{16 \times 12}{5 \times 13} = \frac{192}{65} = 2\tfrac{62}{65}.$$

Now $\dfrac{1}{5x} = \dfrac{13}{16 \times 12}$; $\quad \therefore \dfrac{1}{3y} = \dfrac{1}{4} - \dfrac{13}{192} = \dfrac{48-13}{192}$;

$$\therefore y = \frac{64}{35} = 1\tfrac{29}{35}.$$

19. $$\left.\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 1 - \frac{x}{c} \\ \frac{y}{a} + \frac{x}{b} &= 1 + \frac{y}{c} \end{aligned}\right\}, \text{ find } x \text{ and } y.$$

Then $\left(\dfrac{1}{a} + \dfrac{1}{c}\right)x + \dfrac{1}{b}y = 1$............(1),

$\dfrac{1}{b}x + \left(\dfrac{1}{a} - \dfrac{1}{c}\right)y = 1$............(2).

Multiply (1) by $\dfrac{1}{a} - \dfrac{1}{c}$, (2) by $\dfrac{1}{b}$, and subtract;

$$\therefore \left(\frac{1}{a^2} - \frac{1}{c^2} - \frac{1}{b^2}\right)x = \frac{1}{a} - \frac{1}{c} - \frac{1}{b};$$

$$\therefore x = \frac{abc(ab + ac - bc)}{a^2b^2 + a^2c^2 - b^2c^2}; \text{ similarly, } y = \frac{abc(ac - ab - bc)}{a^2b^2 + a^2c^2 - b^2c^2}.$$

24. $$\left.\begin{aligned} y^{\frac{1}{2}} - (20 - x)^{\frac{1}{2}} &= (y - x)^{\frac{1}{2}}, \ldots\ldots (1) \\ 3(20 - x)^{\frac{1}{2}} &= 2(y - x)^{\frac{1}{2}}, \ldots (2) \end{aligned}\right\}, \text{ find } x \text{ and } y.$$

$$\text{Then } y^{\frac{1}{2}} = \left(\frac{2}{3} + 1\right)(y - x)^{\frac{1}{2}};$$

$$\therefore y = \frac{25}{9}(y - x);$$

$$\therefore 25x = 16y;$$

$$\therefore 9x = 16(y - x).$$

From (2), $$9(20 - x) = 4(y - x) = \frac{9x}{4};$$

$$\therefore 4 \times 20 - 4x = x;$$

$$\therefore x = \frac{80}{5} = 16, \text{ and } y = 25.$$

Ex. 37.

3. $$\left.\begin{aligned} 3x - 7y + 4z &= 1 \\ -5x + 9y - z &= 22 \\ x - 2y + z &= 0 \end{aligned}\right\} \begin{aligned} &(1) \\ &(2), \text{ find } x, y, \text{ and } z. \\ &(3) \end{aligned}$$

Adding (2) and (3) we get, $-4x + 7y = 22 \ldots\ldots\ldots (4)$,

from (1) $3x - 7y + 4z = 1$,

from (3) $4x - 8y + 4z = 0$;

$\therefore -x + y = 1$, or, $-4x + 4y = 4$,

from (4) $-4x + 7y = 22$;

$\therefore 3y = 18$,

and $y = 6$;

$$\therefore x = y - 1 = 5,$$

$$\text{and } z = 2y - x = 7.$$

9. $$\left.\begin{aligned} &\frac{2x + z - 4}{12} + \frac{3y - 6z + 1}{13} = \frac{x - 2}{4}, \\ &\frac{3x - 2y + 5}{5} - \frac{4x - 5y + 7z}{7} = \frac{2}{7} + \frac{3y - 9z + 6}{6}, \\ &\frac{x}{9} - y + 3z = 2. \end{aligned}\right\} \begin{aligned} &(1) \\ &(2) \\ &(3) \end{aligned}$$

From (1), $(26x+13z-52)+(36y-72z+12)=39x-78$,

$$\begin{aligned} \text{whence } 13x-36y+59z&=38,\\ \text{from (3), } \underline{13x-117y+351z}&\underline{=234};\\ \therefore\ 292z-81y&=196\ \text{.........}\ (4). \end{aligned}$$

From (2),

$$(126x-84y+210)-(120x-150y+210z)=60+(105y-315z+210),$$

$$\begin{aligned} \text{whence } 6x-39y+105z&=60;\\ \text{or } 2x-13y+\ \ 35z&=20,\\ \text{and from (3), } \underline{2x-18y+\ \ 54z}&\underline{=36};\\ \therefore\ 19z-5y&=16; \end{aligned}$$

$$\begin{aligned} \therefore\ 19\times 292z-1460y&=4672,\\ \text{and from (4), } \underline{19\times 292z-1539y}&\underline{=3724};\\ \therefore\ 79y=948,\ \text{or } y&=12; \end{aligned}$$

$$\therefore\ z=\frac{16+5y}{19}=4,\ \text{and } x=18+9y-27z=18.$$

13. $xy=3(x+y)$, $xz=8(x+z)$, $7yz=9(y+z)$, find x, y, z.

Divide the first by $3xy$,

$$\text{then } \frac{1}{x}+\frac{1}{y}=\frac{1}{3}\ \text{............}(1),$$

$$\text{similarly } \frac{1}{x}+\frac{1}{z}=\frac{1}{8}\ \text{............}(2),$$

$$\text{and } \frac{1}{y}+\frac{1}{z}=\frac{7}{9}\ \text{............}(3).$$

Add and divide by 2,

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{8}+\frac{7}{9}\right).$$

From this subtract (3),

$$\text{then } \frac{1}{x}=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{8}-\frac{7}{9}\right)=-\frac{23}{144};\quad \therefore\ x=-6\tfrac{6}{23}.$$

Similarly,

$$\text{subtract (2), } \frac{1}{y}=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{8}+\frac{7}{9}\right)=\frac{71}{144};\qquad \therefore\ y=2\tfrac{2}{71}.$$

$$\text{.........}\ (1),\ \frac{1}{z}=\frac{1}{2}\left(-\frac{1}{3}+\frac{1}{8}+\frac{7}{9}\right)=\frac{41}{144};\quad \therefore\ z=3\tfrac{21}{41}.$$

Ex. 38.

20. Let x = the time A took to do the work,

y = ... B ....,

z = ... C ....,

and let w = the work.

Then $\frac{w}{x} = A$'s daily work; $\frac{w}{y} = B$'s; $\frac{w}{z} = C$'s,

and since B and C working together can perform m times as much per day as A,

$$\frac{mw}{x} = \frac{w}{y} + \frac{w}{z}; \quad \therefore \frac{m}{x} = \frac{1}{y} + \frac{1}{z} \quad \text{.............(1).}$$

$$\text{Similarly } \frac{n}{y} = \frac{1}{x} + \frac{1}{z} \quad \text{.............(2),}$$

$$\text{and } \frac{p}{z} = \frac{1}{x} + \frac{1}{y} \quad \text{.............(3).}$$

From (1) and (2), $\frac{m}{x} - \frac{n}{y} = \frac{1}{y} - \frac{1}{x}$;

$$\therefore (m+1)\frac{1}{x} = (n+1)\frac{1}{y};$$

$$\text{or } \frac{x}{y} = \frac{m+1}{n+1} = \frac{\text{time taken by } A}{\text{time taken by } B}.$$

$$\text{Similarly } \frac{x}{z} = \frac{m+1}{p+1} = \frac{\text{time taken by } A}{\text{time taken by } C},$$

$$\text{and } \frac{y}{z} = \frac{n+1}{p+1} = \frac{\text{time taken by } B}{\text{time taken by } C}.$$

Again from (1), $m = \frac{x}{y} + \frac{x}{z} = \frac{m+1}{n+1} + \frac{m+1}{p+1}$;

$$\therefore \frac{1}{n+1} + \frac{1}{p+1} = \frac{m}{m+1} = 1 - \frac{1}{m+1};$$

$$\therefore \frac{1}{m+1} + \frac{1}{n+1} + \frac{1}{p+1} = 1.$$

23. Let the length of the course be x yards, and the duration of the race be t minutes.

Then A runs $x + 90$ yards, whilst B runs $x - 90$ yards, and A would run $2x + \frac{1}{6}x$ yards, whilst B would run $2x - \frac{1}{6}x$.

Hence $\frac{x+90}{x-90} = \frac{A\text{'s rate of running}}{B\text{'s} \ldots\ldots\ldots\ldots} = \frac{2x+\frac{1}{6}x}{2x-\frac{1}{6}x} = \frac{12+1}{12-1}$;

$$\therefore \frac{x}{90} = \frac{12}{1}; \quad \therefore x = 1080 \text{ yards.}$$

Again, $\frac{A\text{'s time of running}}{B\text{'s} \ldots\ldots\ldots\ldots} = \frac{B\text{'s rate of running}}{A\text{'s} \ldots\ldots\ldots\ldots}$;

$$\therefore \frac{t}{t+3} = \frac{2x-\frac{1}{6}x}{2x+\frac{1}{6}x} = \frac{11}{13}.$$

Dividendo, $\frac{t}{3} = \frac{11}{2}$,

$$\therefore t = \frac{33}{2} = 16\tfrac{1}{2} \text{ minutes.}$$

25. Let x = the No. of bushels of wheat,

y = ... barley,

z = ... rye,

then $x + y + z = 100$ ....(1).

The value in £'s is therefore

$$\frac{x}{5} + \frac{7y}{60} + \frac{3z}{20} = 100 \times \frac{1}{6} \ldots\ldots\ldots\ldots(2),$$

and $2z \times \frac{3}{20} + \frac{7y}{60} + (x+10)\frac{1}{5} = \frac{1}{6}(x+y+2z+10) \ldots(3)$,

from (2), $12x + 7y + 9z = 1000$,

from (1), $9x + 9y + 9z = 900$;

$$\therefore 3x - 2y = 100 \ldots\ldots\ldots\ldots\ldots(4).$$

Again from (3),

$$18z + 7y + 12x + 120 = 10x + 10y + 20z + 100;$$

$$\therefore 2x - 3y - 2z = -20,$$

from (1), $2x + 2y + 2z = 200$;

$$\therefore 4x - y = 180, \text{ or } 8x - 2y = 360,$$

from (4), $3x - 2y = 100$;

$$\therefore 5x = 260, \; x = 52;$$

$$\therefore\ y = 4x - 180 = 28,$$
$$z = 100 - (x+y) = 20.$$

Ex. 39.

6. $$3\left(\frac{x^2-9}{x^2+3}\right) + 4\left(\frac{22\frac{1}{2}+x^2}{x^2+9}\right) = 7;$$

$$\therefore\ 3\left(\frac{x^2-9}{x^2+3}\right) - 3 + 4\left(\frac{22\frac{1}{2}+x^2}{x^2+9}\right) - 4 = 0,$$

whence $\dfrac{54}{x^2+9} = \dfrac{36}{x^2+3}$, or $\dfrac{3}{x^2+9} = \dfrac{2}{x^2+3}$;

$$\therefore\ 3x^2 + 9 = 2x^2 + 18;$$
$$\therefore\ x^2 = 9, \text{ and } x = \pm 3.$$

10. $$\frac{35-2x}{9} + \frac{5x^2+7}{5x^2-7} = \frac{17 - \frac{2}{3}x}{3}, \text{ find } x;$$

$$\therefore\ \frac{5x^2+7}{5x^2-7} = \frac{51-2x}{9} - \frac{35-2x}{9} = \frac{16}{9}.$$

Componendo and Dividendo, $\dfrac{5x^2}{7} = \dfrac{25}{7}$;

$$\therefore\ x = \pm\sqrt{5}.$$

17. $$\frac{1+x^3}{(1+x)^2} + \frac{1-x^3}{(1-x)^2} = a,$$

or $$\frac{1-x+x^2}{1+x} + \frac{1+x+x^2}{1-x} = a;$$

$$\therefore\ \frac{1-x+x^2}{1+x} - x + \frac{1+x+x^2}{1-x} + x = a,$$

or $$\frac{1-2x}{1+x} + \frac{1+2x}{1-x} = a,$$

whence $x = \pm\left(\dfrac{a-2}{a+4}\right)^{\frac{1}{2}}$.

21. $$\frac{(1+x)^{\frac{1}{2}}-1}{(1-x)^{\frac{1}{2}}+1} + \frac{(1-x)^{\frac{1}{2}}+1}{(1+x)^{\frac{1}{2}}-1} = a, \text{ find } x.$$

Then

$$\frac{(1+x)^{\frac{1}{2}}-1}{(1-x)^{\frac{1}{2}}+1}\times\frac{(1-x)^{\frac{1}{2}}-1}{(1-x)^{\frac{1}{2}}-1}+\frac{(1-x)^{\frac{1}{2}}+1}{(1+x)^{\frac{1}{2}}-1}\times\frac{(1+x)^{\frac{1}{2}}+1}{(1+x)^{\frac{1}{2}}+1}=a;$$

$$\therefore\ \frac{(1-x^2)^{\frac{1}{2}}+1-\{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{1}{2}}\}}{-x}$$

$$+\frac{(1-x^2)^{\frac{1}{2}}+1+\{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{1}{2}}\}}{x}=a;$$

$$\therefore\ 2\{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{1}{2}}\}=ax,$$

$$4\{2+2(1-x^2)^{\frac{1}{2}}\}=a^2x^2,$$

$$64(1-x^2)=(a^2x^2-8)^2=a^4x^4-16a^2x^2+64;$$

$$\therefore\ 16x^2(a^2-4)=a^4x^4;$$

$$\therefore\ x=\pm\frac{4}{a^2}(a^2-4)^{\frac{1}{2}}.$$

22. $$\{(1+x)^2-ax\}^{\frac{1}{2}}+\{(1-x)^2+ax\}^{\frac{1}{2}}=x.$$

Transpose, and square,

$$1+2x+x^2-ax=x^2+1-2x+x^2+ax-2x\{(1-x)^2+ax\}^{\frac{1}{2}};$$

$$\therefore\ 4x-x^2-2ax=-2x\{(1-x)^2+ax\}^{\frac{1}{2}}.$$

Divide by x, and square,

$$x^2+4ax+4a^2-8x-16a+16=4-8x+4x^2+4ax;$$

$$\therefore\ 3x^2=4a^2+12-16a;$$

$$\therefore\ x^2=\frac{4}{3}(a^2+3-4a)=\frac{4(a-1)(a-3)}{3};$$

$$\therefore\ x=\pm 2\left\{\frac{(a-1)(a-3)}{3}\right\}^{\frac{1}{2}}.$$

Ex. 40.

9. $$x^2-\frac{9}{5}x+\frac{9}{20}=0;$$

$$\therefore\ x^2-\frac{9}{5}x+\frac{81}{100}=\frac{81}{100}-\frac{9}{20}=\frac{36}{100};$$

$$\therefore\ x - \frac{9}{10} = \pm\frac{6}{10};$$

$$\therefore\ x = \frac{3}{2},\ \text{or}\ \frac{3}{10}.$$

17. $17x^2 + 19x = 1848$, find x.

By the Hindu method,

(1) Multiply the equation by $4 \times$ coefficient of x^2.

(2) Add to both sides the square of the coefficient of x.

Then

$$4 \times 17^2x^2 + 4 \times 17 \times 19x + 19^2 = 4 \times 17 \times 1848 + 19^2 = 126025;$$

$$\therefore\ 2 \times 17x + 19 = \pm 355,$$

$$x = \frac{\pm 355 - 19}{34} = 9\tfrac{15}{17},\ \text{or} - 11.$$

48. $$x + (x^2 - ax + b^2)^{\frac{1}{2}} = \frac{x^2}{a} + b;$$

$$\therefore\ (x^2 - ax + b^2) - a(x^2 - ax + b^2)^{\frac{1}{2}} = b^2 - ab;$$

$$\therefore\ (x^2 - ax + b^2) - a(x^2 - ax + b^2)^{\frac{1}{2}} + \frac{a^2}{4} = b^2 - ab + \frac{a^2}{4};$$

$$\therefore\ (x^2 - ax + b^2)^{\frac{1}{2}} - \frac{a}{2} = \pm\left(b - \frac{a}{2}\right),$$

whence $x^2 - ax + b^2 = b^2$, or $(a-b)^2$.

(1°) $$x^2 - ax + b^2 = b^2;$$

$$\therefore\ x^2 = ax;$$

$$\therefore\ x = a,\ 0.$$

(2°) $$x^2 - ax + b^2 = a^2 - 2ab + b^2;$$

$$\therefore\ x^2 - ax + \frac{a^2}{4} = \frac{5a^2 - 8ab}{4};$$

$$\therefore\ x - \frac{a}{2} = \pm\frac{a}{2}\left(5 - \frac{8b}{a}\right)^{\frac{1}{2}};$$

$$\therefore\ x = \frac{a}{2}\left\{1 \pm \left(5 - \frac{8b}{a}\right)^{\frac{1}{2}}\right\}.$$

52. $\{x+(2x-1)^{\frac{1}{2}}\}^{\frac{1}{2}}-\{x-(2x-1)^{\frac{1}{2}}\}^{\frac{1}{2}}=\dfrac{3}{5}\left\{\dfrac{10x}{x+(2x-1)^{\frac{1}{2}}}\right\}^{\frac{1}{2}};$

$$\therefore\ \{x+(2x-1)^{\frac{1}{2}}\}-\{(x-1)^2\}^{\frac{1}{2}}=\frac{3}{5}(10x)^{\frac{1}{2}};$$

$$\therefore\ x+(2x-1)^{\frac{1}{2}}-x+1=3\left(\frac{2x}{5}\right)^{\frac{1}{2}};$$

$$\therefore\ 2x-1=\frac{18x}{5}-6\left(\frac{2x}{5}\right)^{\frac{1}{2}}+1;$$

$$\therefore\ \frac{8x}{5}+2=6\left(\frac{2x}{5}\right)^{\frac{1}{2}};$$

$$\therefore\ \frac{2x}{5}-\frac{3}{2}\left(\frac{2x}{5}\right)^{\frac{1}{2}}+\frac{9}{16}=-\frac{1}{2}+\frac{9}{16}=\frac{1}{16};$$

$$\left(\frac{2x}{5}\right)^{\frac{1}{2}}=\frac{3}{4}\pm\frac{1}{4}=1, \text{ or } \frac{1}{2};$$

$$\therefore\ x=\frac{5}{2}, \text{ or } \frac{5}{8}.$$

54. $cx=\{(1+x)^{\frac{1}{2}}-1\}.\{(1-x)^{\frac{1}{2}}+1\};$

$$\therefore\ c\{(1+x)-1\}=\{(1+x)^{\frac{1}{2}}-1\}.\{(1-x)^{\frac{1}{2}}+1\}....(1);$$

$$\therefore\ c\{(1+x)^{\frac{1}{2}}+1\}=(1-x)^{\frac{1}{2}}+1;$$

$$\therefore\ c(1+x)^{\frac{1}{2}}-(1-x)^{\frac{1}{2}}=1-c;$$

$$\therefore\ c^2(1+x)+(1-x)-2c(1-x^2)^{\frac{1}{2}}=1-2c+c^2;$$

$$\therefore\ 2c(1-x^2)^{\frac{1}{2}}=2c+(c^2-1)x;$$

$$\therefore\ 4c^2-4c^2x^2=4c^2+c^4x^2-2c^2x^2+x^2+4c(c^2-1)x;$$

$$\therefore\ x^2\{c^4+2c^2+1\}+4c(c^2-1)x=0;$$

$$\therefore\ x^2+\frac{4c(c^2-1)}{(c^2+1)^2}x=0;$$

$$\therefore\ x=0, \text{ and } \frac{4c(1-c^2)}{(1+c^2)^2}.$$

55. $\dfrac{x}{8}+\dfrac{1}{2x}=\left(\dfrac{x}{3}+\dfrac{1}{4}\right)^{\frac{1}{2}}-\dfrac{2}{3}$, find x.

Then $\left(\dfrac{x^2}{64}+\dfrac{1}{8}+\dfrac{1}{4x^2}\right)+\dfrac{4}{3}\left(\dfrac{x}{8}+\dfrac{1}{2x}\right)+\dfrac{4}{9}=\dfrac{x}{3}+\dfrac{1}{4};$

$$\therefore \left(\frac{x^2}{64}-\frac{1}{8}+\frac{1}{4x^2}\right)-\frac{4}{3}\left(\frac{x}{8}-\frac{1}{2x}\right)+\frac{4}{9}=0;$$

$$\therefore \frac{x}{8}-\frac{1}{2x}=\frac{2}{3};$$

whence $x=6$, or $-\dfrac{2}{3}$.

57. $$\frac{x^2}{4}=\frac{x-12}{x^2-18};$$

$$\therefore x^4-18x^2=4x-48;$$

$$\therefore x^4-14x^2+49=4x^2+4x+1;$$

$$\therefore x^2-7=\pm(2x+1);$$

whence $x=4,\ -2,\ -1\pm\sqrt{7}$.

61. $4x^2+12x(1+x)^{\frac{1}{2}}=27(1+x)$, find x.

Then $4x^2+4x\times 3(1+x)^{\frac{1}{2}}+9(1+x)=36(1+x)$;

$$\therefore 2x=(\pm 6-3)(1+x)^{\frac{1}{2}}=3(1+x)^{\frac{1}{2}},\text{ or }-9(1+x)^{\frac{1}{2}};$$

whence $x=3,\ -\dfrac{3}{4},\ \dfrac{9}{8}(9\pm\sqrt{97})$.

62. $$2x^2+(x^2+9)^{\frac{1}{2}}=x^4-9.$$

Then $(x^2+9)+(x^2+9)^{\frac{1}{2}}+\dfrac{1}{4}=x^4-x^2+\dfrac{1}{4}$;

whence $x=\pm\left(\dfrac{3\pm\sqrt{41}}{2}\right)^{\frac{1}{2}},\ \pm\left(\dfrac{1+\sqrt{37}}{2}\right)^{\frac{1}{2}}$.

63. $(a+x)^{\frac{2}{3}}+4(a-x)^{\frac{2}{3}}-5(a^2-x)^{\frac{1}{3}}=0$.

Then $\left(\dfrac{a+x}{a-x}\right)^{\frac{2}{3}}-5\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{3}}+\dfrac{25}{4}=\dfrac{25}{4}-4=\dfrac{9}{4}$;

$$\therefore \frac{a+x}{a-x}=\left(\frac{5}{2}\pm\frac{3}{2}\right)^3=64,\text{ or }1;$$

$$\therefore x=\frac{63}{65}a,\ -0.$$

64. $$\frac{a+x}{(a-x)^{\frac{1}{2}}} + \frac{a-x}{(a+x)^{\frac{1}{2}}} = 2a^{\frac{1}{2}}.$$

Squaring, $$\frac{(a+x)^2}{a-x} + 2(a^2-x^2)^{\frac{1}{2}} + \frac{(a-x)^2}{a+x} = 4a;$$

$$\therefore \frac{(a+x)^3+(a-x)^3}{a^2-x^2} = 4a - 2(a^2-x^2)^{\frac{1}{2}};$$

$$\therefore a^3 + 3ax^2 = 2a^3 - 2ax^2 - (a^2-x^2)^{\frac{3}{2}}.$$

$$\therefore (a^2-x^2)^{\frac{3}{2}} = a^3 - 5ax^2 = 5a(a^2-x^2) - 4a^3.$$

Assume $a^2 - x^2 = y^2$,

then $y^3 - 5ay^2 = -4a^3$,

or $y^2(y-a) = 4a(y^2-a^2)$;

$\therefore y - a = 0$, or $y = a$;

$\therefore x = 0$.

Again, $y^2 = 4ay + 4a^2$;

$$\therefore y = 2a(1 \pm \sqrt{2}); \quad \therefore x = \pm a(\pm 8\sqrt{2} - 11)^{\frac{1}{2}}.$$

65. $$(1+x)^{\frac{2}{5}} + (1-x)^{\frac{2}{5}} = (1-x^2)^{\frac{1}{5}}.$$

Dividing by $(1+x)^{\frac{2}{5}}$, and transposing, we have

$$\left(\frac{1-x}{1+x}\right)^{\frac{2}{5}} - \left(\frac{1-x}{1+x}\right)^{\frac{1}{5}} = -1.$$

Completing the square, $$\left(\frac{1-x}{1+x}\right)^{\frac{2}{5}} - \left(\frac{1-x}{1+x}\right)^{\frac{1}{5}} + \frac{1}{4} = -\frac{3}{4},$$

whence $$\left(\frac{1-x}{1+x}\right)^{\frac{1}{5}} = \frac{1}{2}(1 \pm \sqrt{-3});$$

$$\therefore \frac{1-x}{1+x} = \left\{\frac{1}{2}(1 \pm \sqrt{-3})\right\}^5 = \frac{\{1 \pm \sqrt{-3}\}^5}{2^5},$$

whence $$x = \frac{(1 \pm \sqrt{-3})^5 - 2^5}{(1 \pm \sqrt{-3})^5 + 2^5} = \frac{1 \pm \sqrt{-3}}{-3 \pm \sqrt{-3}} = \pm \frac{1}{3}\sqrt{-3}.$$

67. $$\frac{1+x^3}{(1+x)^3}+\frac{1-x^3}{(1-x)^3}=c, \quad \text{find } x.$$

Then $$\frac{1-x+x^2}{(1+x)^2}+\frac{1+x+x^2}{(1-x)^2}=c;$$

$$\therefore \frac{3x}{(1-x)^2}-\frac{3x}{(1+x)^2}=c-2;$$

$$\therefore \frac{4x^2}{(1-x^2)^2}=\frac{c-2}{3}.$$

Componendo, $$\frac{(1+x^2)^2}{(1-x^2)^2}=\frac{c+1}{3}.$$

Whence $$x=\left\{\frac{\sqrt{c+1}\mp\sqrt{3}}{\sqrt{c+1}\pm\sqrt{3}}\right\}^{\frac{1}{2}}=\left\{\frac{c+4\pm 2\sqrt{3(c+1)}}{c-2}\right\}^{\frac{1}{2}}.$$

68. $$(a+x)^{\frac{1}{4}}+(a-x)^{\frac{1}{4}}=h;$$

$$\therefore a+x+a-x+4(a^2-x^2)^{\frac{1}{4}}\{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}+2(a^2-x^2)^{\frac{1}{4}}\}$$
$$-2(a^2-x^2)^{\frac{1}{2}}=h^4;$$

$$\therefore 2a+4h^2(a^2-x^2)^{\frac{1}{4}}-2(a^2-x^2)^{\frac{1}{2}}=h^4;$$

$$\therefore (a^2-x^2)^{\frac{1}{2}}-2h^2(a^2-x^2)^{\frac{1}{4}}=a-\frac{h^4}{2}.$$

Completing the square,

$$(a^2-x^2)^{\frac{1}{2}}-2h^2(a^2-x^2)^{\frac{1}{4}}+h^4=a+\frac{h^4}{2},$$

whence $$(a^2-x^2)^{\frac{1}{4}}-h^2=\pm\left(a+\frac{h^4}{2}\right)^{\frac{1}{2}};$$

$$\therefore (a^2-x^2)=\left\{h^2\pm\left(a+\frac{h^4}{2}\right)^{\frac{1}{2}}\right\}^4;$$

$$\therefore x^2=a^2-\left\{h^2\pm\left(a+\frac{h^4}{2}\right)^{\frac{1}{2}}\right\}^4,$$

and $$x=\pm\left[a^2-\left\{h^2\pm\left(a+\frac{h^4}{2}\right)^{\frac{1}{2}}\right\}^4\right]^{\frac{1}{2}}.$$

69. $$2x(1-x^4)^{\frac{1}{2}}=a(1+x^4), \quad \text{find } x.$$

Then $$2\left(\frac{1}{x^2}-x^2\right)^{\frac{1}{2}}=a\left(\frac{1}{x^2}+x^2\right).$$

Square and transpose,

$$a^2\left(x^2+\frac{1}{x^2}\right)^2+4\left(x^2-\frac{1}{x^2}\right)=0.$$

Subtract $4a^2$ from both sides, and complete the square,

$$a^2\left(x^2-\frac{1}{x^2}\right)^2+4\left(x^2-\frac{1}{x^2}\right)+\frac{4}{a^2}=\frac{4}{a^2}-4a^2=\frac{4}{a^2}(1-a^4);$$

$$\therefore\ x^2-\frac{1}{x^2}=\frac{2}{a^2}\left(-1\pm\sqrt{1-a^4}\right).$$

Whence $x=\pm\frac{1}{a}\left[-1\pm(1-a^4)^{\frac{1}{2}}\pm\{2\pm2(1-a^4)^{\frac{1}{2}}\}^{\frac{1}{2}}\right]^{\frac{1}{2}}$.

Ex. 41.

9. $$x^4+x^3-4x^2+x+1=0;$$

$$\therefore\ (x^4+1)+x(x^2+1)=4x^2;$$

$$\therefore\ (x^2+1)^2+x(x^2+1)=6x^2;$$

$$\therefore\ (x^2+1)^2+x(x^2+1)+\frac{x^2}{4}=\frac{25x^2}{4};$$

$$\therefore\ x^2+1+\frac{x}{2}=\pm\frac{5x}{2}.$$

Whence the values of x are found to be

$$1,\ 1,\ \frac{1}{2}\{-3\pm\sqrt{5}\}.$$

10. $$x^4-x^3+\frac{5}{4}x^2-x+1=0;$$

$$\therefore\ (x^4+2x^2+1)-x(x^2+1)=-\frac{5}{4}x^2+2x^2=\frac{3}{4}x^2;$$

$$\therefore\ (x^2+1)^2-x(x^2+1)+\frac{x^2}{4}=x^2,$$

$$\text{whence}\ x^2+1-\frac{x}{2}=\pm x;$$

$$\therefore\ x=\frac{1}{4}\{3\pm\sqrt{-7}\},\ \text{or}=\frac{1}{4}(-1\pm\sqrt{-15}).$$

15. $$\frac{1+x^5}{(1+x)^5}=a,\ \text{or}\ \frac{(1+x)(1-x+x^2-x^3+x^4)}{(1+x)(1+4x+6x^2+4x^3+x^4)}=a;$$

$$\therefore\ x=-1\ \text{is a root.}$$

Also $\quad 1 - x + x^2 - x^3 + x^4 = a(1 + 4x + 6x^2 + 4x^3 + x^4)$.

Whence $\quad (a-1)(x^4+1) + (4a+1)(x^3+x) + (6a-1)x^2 = 0$,

$$\text{or } (a-1)\left(x+\frac{1}{x}\right)^2 + (4a+1)\left(x+\frac{1}{x}\right) = -(4a+1),$$

$$\text{whence } x + \frac{1}{x} = \frac{(4a+1)^{\frac{1}{2}}}{2(a-1)} \cdot \{-(4a+1)^{\frac{1}{2}} \pm 5^{\frac{1}{2}}\}$$

$$= 2p \text{ suppose};$$

$$\therefore\ x^2 - 2px = -1,$$

$$x = p \pm \sqrt{p^2-1};$$

$$\therefore\ x = \frac{(1+4a) + 5^{\frac{1}{2}}(1+4a)^{\frac{1}{2}} \pm \{-10 + 60a \pm 2 \times 5^{\frac{1}{2}}(1+4a)^{\frac{3}{2}}\}^{\frac{1}{2}}}{4(1-a)}.$$

Ex. 42.

27. $$\left.\begin{array}{l} x^{\frac{1}{2}} + y^{\frac{1}{2}} = 4, \ldots\ldots(1) \\ x^{\frac{3}{2}} + y^{\frac{3}{2}} = 28, \ldots(2) \end{array}\right\},\ \text{find } x \text{ and } y.$$

Dividing (2) by (1), $\quad x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 7$,

squaring (1), $\quad x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 16$;

$$\therefore\ 3x^{\frac{1}{2}}y^{\frac{1}{2}} = 9, \text{ or } x^{\frac{1}{2}}y^{\frac{1}{2}} = 3 \ldots\ldots(3).$$

From (1), $\quad x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 16$,

from (3), $\quad 4x^{\frac{1}{2}}y^{\frac{1}{2}} = 12$;

$$\therefore\ x - 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 4,$$

$$\text{or } x^{\frac{1}{2}} - y^{\frac{1}{2}} = \pm 2:$$

$$\text{but } x^{\frac{1}{2}} + y^{\frac{1}{2}} = 4;$$

$$\therefore\ x^{\frac{1}{2}} = 3, \text{ or } 1; \quad x = 9, \text{ or } 1;$$

$$y^{\frac{1}{2}} = 1, \text{ or } 3; \quad y = 1, \text{ or } 9.$$

37. $x^4 - y^4 = 14560, \quad x - y = 8,$ find x and y.

Substitute $x - 8$ for y in (1).

Then $x^4 - (x^4 - 4 \times 8x^3 + 6 \times 8^2x^2 - 4 \times 8^3x + 8^4) = 14560$;

$$\therefore 4 \times 8x^3 - 6 \times 8^2x^2 + 4 \times 8^3x = 14560 + 4096 = 18656;$$

$$\therefore x^3 - 12x^2 + 64x - 583 = 0,$$

$$\text{or} \quad x(x^2 - 12x + 11) + 53(x - 11) = 0;$$

$$\therefore x - 11 = 0, \quad \text{and} \quad x(x - 1) + 53 = 0.$$

Whence $\left.\begin{array}{l} x = 11 \\ y = 3 \end{array}\right\}$, and $\begin{array}{l} x = \frac{1}{2}\{1 \pm \sqrt{-211}\}, \\ y = \frac{1}{2}\{-15 \pm \sqrt{-211}\}. \end{array}$

38. $x^4 + y^4 = 641, \quad xy(x^2 + y^2) = 290,$ find x and y.

Then $\left(\dfrac{290}{xy}\right)^2 - 2x^2y^2 = x^4 + y^4 = 641$;

$$\therefore 2(xy)^4 + 641(xy)^2 = 290^2,$$

$$4 \times 2^2(xy)^4 + 4 \times 2 \times 641(xy)^2 + 641^2 = 641^2 + 4 \times 2 \times 290^2 = 1041^2;$$

$$\therefore 2x^2y^2 = \pm \frac{1041 - 641}{2} = 200, \text{ or } -841.$$

Hence $x^2 + y^2 = 29, \quad$ or $\pm 10\sqrt{-2}$,

$x^2 - y^2 = 21, \quad$ or $\pm \sqrt{1482}$;

$$\therefore x = \pm 5, \ \pm 2, \ \pm\{\pm 5\sqrt{-2} \pm \sqrt{370 \cdot 5}\}^{\frac{1}{2}},$$

$$y = \pm 2, \ \pm 5, \ \pm\{\pm 5\sqrt{-2} \mp \sqrt{370 \cdot 5}\}^{\frac{1}{2}}.$$

40. $x^2 + z^2 = 3xz, \quad x^5 - z^5 = 2,$ find x and z;

$$\therefore x + z = \pm(5xz)^{\frac{1}{2}}, \quad x^4 - x^3z + x^2z^2 - xz^3 + z^4 = \pm \frac{2}{(5xz)^{\frac{1}{2}}}.$$

But $x^4 \qquad + x^2z^2 \qquad + z^4 = 8x^2z^2$;

$$\therefore xz(x^2 + z^2), \text{ or } 3x^2z^2 = 8x^2z^2 \mp \frac{2}{(5xz)^{\frac{1}{2}}};$$

$$\text{Hence}\quad 5x^2z^2 = \pm \frac{2}{(5xz)^{\frac{1}{2}}};$$

$$\therefore\ (xz)^{\frac{5}{2}} = \pm \frac{2}{5^{\frac{3}{2}}};$$

$$\therefore\ xz = \left(\frac{4}{125}\right)^{\frac{1}{5}} = \left(\frac{32}{1000}\right)^{\frac{1}{5}} = \cdot 2\ (100)^{\frac{1}{5}}.$$

$$\text{Then}\quad x + z = \pm (100)^{\frac{1}{10}},$$

$$x - z = \pm \{\cdot 2\, (100)^{\frac{1}{5}}\}^{\frac{1}{2}};$$

$$\therefore\ x = \pm (10)^{\frac{1}{5}} (\cdot 5 + \cdot 1 \times 5^{\frac{1}{2}});\quad z = \pm (10)^{\frac{1}{5}} (\cdot 5 - \cdot 1 \times 5^{\frac{1}{2}}).$$

41. $x - y = 1$, $(x^2 + y^2)(x^3 - y^3) = 247$, find x and y.

Now $x^2 + y^2 = 1 + 2xy$, by squaring (1),

and $x^3 - y^3 = 1 + 3xy(x - y) = 1 + 3xy$, by cubing (1).

$$\text{Hence}\quad (1 + 2xy)(1 + 3xy) = 247;$$

$$\therefore\ xy = 6, \text{ or } -\frac{41}{6};$$

$$\therefore\ x + y = \pm 5, \text{ or } \pm\left(-\frac{79}{3}\right)^{\frac{1}{2}};$$

$$\therefore\ x = 3,\ -2,\ \frac{1}{2}\left\{\pm\left(-\frac{79}{3}\right)^{\frac{1}{2}} + 1\right\},$$

$$y = 2,\ -3,\ \frac{1}{2}\left\{\pm\left(-\frac{79}{3}\right)^{\frac{1}{2}} - 1\right\}.$$

42. $x^3y^2 + xy^4 = 156$, $2x^3y^2 - x^2y^3 = 144$, find x and y.

The equations being homogeneous, put $y = xz$;

$$\text{Then}\quad \frac{z^2 + z^4}{2z^2 - z^3} = \frac{156}{144}, \text{ or } \frac{1 + z^2}{2 - z} = \frac{13}{12}.$$

$$\text{Hence}\quad z = \frac{2}{3}, \text{ or } -\frac{7}{4}.$$

$$\text{Now } x^2y^2(2x-y)=144,$$

$$\text{and } \frac{2x-y}{x}=2-z=2-\frac{2}{3}=\frac{4}{3}, \text{ or } 2+\frac{7}{4}=\frac{15}{4};$$

$$\therefore x^3y^2=\frac{3\times 144}{4}=108, \text{ or } \frac{4\times 144}{15};$$

$$\therefore xy^4=156-108=48.$$

$$\text{But } \frac{y}{x}=\frac{2}{3};$$

$$\therefore y^5=\frac{2}{3}\times 48=32;$$

$$\therefore y=2, \text{ and } x=3.$$

$$x^2y^3=\frac{8}{15}\times 144-144=-\frac{7}{15}\times 144,$$

$$\text{and } \frac{y^2}{x^2}=\frac{49}{16};$$

$$\therefore y^5=-\frac{7^3}{15}\times 9=-\frac{1029}{5};$$

$$\therefore y=-(205\cdot 8)^{\frac{1}{5}}, \text{ and } x=\frac{4}{7}(205\cdot 8)^{\frac{1}{5}}.$$

43. $$\left.\begin{aligned} \frac{y}{x}-\frac{81}{xy}&=(2y+18)\frac{x^{\frac{1}{2}}}{y} \\ y+3x^{\frac{3}{2}}&=9+3(x^3y)^{\frac{1}{2}} \end{aligned}\right\}, \text{ find } x \text{ and } y.$$

Clearing (1) of fractions,

$$y^2-81=2x^{\frac{3}{2}}(y+9);$$

$$\therefore y+9=0, \text{ or } y=-9,$$

$$\text{and } y-9=2x^{\frac{3}{2}},$$

$$\text{from (2), } y-9=3x^{\frac{3}{2}}(y^{\frac{1}{2}}-1);$$

$$\therefore 3x^{\frac{3}{2}}(y^{\frac{1}{2}}-1)=2x^{\frac{3}{2}};$$

$$\therefore x^{\frac{3}{2}}=0, \text{ or } x=0,$$

$$\text{and } y^{\frac{1}{2}}-1=\frac{2}{3}; \text{ whence } y=\frac{25}{9}, \text{ or } -9.$$

$$\text{But } 2x^{\frac{3}{2}}=y-9=-\frac{56}{9}, \text{ or } -18;$$

$$\therefore x=\left(-\frac{28}{9}\right)^{\frac{2}{3}}, \text{ or } (-9)^{\frac{2}{3}}$$

$$=\frac{2}{3}\left(\frac{98}{3}\right)^{\frac{1}{3}}, \text{ or } 3(3)^{\frac{1}{3}}.$$

44. $$\left.\begin{array}{l}\left(\dfrac{x}{y}\right)^{\frac{1}{2}}+\left(\dfrac{y}{x}\right)^{\frac{1}{2}}=\dfrac{61}{(xy)^{\frac{1}{2}}}+1\\ (x^3y)^{\frac{1}{4}}+(xy^3)^{\frac{1}{4}}=78\end{array}\right\};\ \text{find } x \text{ and } y.$$

$$\text{From (2), } (xy)^{\frac{1}{4}}(\sqrt{x}+\sqrt{y})=78;$$

$$\therefore \sqrt{xy}\,(x+y+2\sqrt{xy})=78^2.$$

$$\text{From (1), } x+y=61+\sqrt{xy};$$

$$\therefore \sqrt{xy}\,(61+3\sqrt{xy})=78^2;$$

$$\text{whence } \sqrt{xy}=36, \text{ or } -\frac{169}{3};$$

$$\text{and } \therefore x=81,\ 16,\ \frac{1}{3}(7+36\sqrt{-22}),$$

$$y=16,\ 81,\ \frac{1}{3}(7-36\sqrt{-22}).$$

45. $$\frac{x}{y}-\frac{y}{x}=\frac{x+y}{x^2+y^2},\quad \frac{x^2}{y^2}-\frac{y^2}{x^2}=\frac{x-y}{y^2};\ \text{find } x \text{ and } y.$$

Divide the 2nd by the 1st, then

$$\frac{x}{y}+\frac{y}{x}=\frac{x-y}{x+y}\cdot\frac{x^2+y^2}{y^2}.$$

$$\text{Put } x=yz;$$

$$\text{then } z+\frac{1}{z}=\frac{z-1}{z+1}(z^2+1);$$

$$\therefore z^2+1=0, \text{ and } z+1=z^2-z.$$

$$\text{The latter gives } z \text{ or } \frac{x}{y}=1\pm\sqrt{2}.$$

$$\text{Now } \frac{\dfrac{x}{y}+1}{y\left(\dfrac{x^2}{y^2}+1\right)}=\frac{x}{y}-\frac{y}{x};$$

$$\therefore \frac{2\pm\sqrt{2}}{y(4\pm2\sqrt{2})}=1\pm\sqrt{2}-\frac{1}{1\pm\sqrt{2}}=\frac{2\pm2\sqrt{2}}{1\pm\sqrt{2}};$$

$$\therefore \frac{1}{2y}=2;\ \text{hence } y=\frac{1}{4}, \text{ and } x=\frac{1\pm\sqrt{2}}{4}.$$

46. $$\left.\begin{array}{l}\dfrac{x^2}{y^2}+\dfrac{2x+y}{y^{\frac{1}{2}}}=20-\dfrac{y^2+x}{y},\\ x-4y+8=0,\end{array}\right\}\text{ find } x \text{ and } y.$$

From the 1st, $x^2+2xy^{\frac{3}{2}}+y^{\frac{5}{2}}=20y^2-(y^3+xy)$;

$$\therefore\ (x^2+2xy^{\frac{3}{2}}+y^3)+y(x+y^{\frac{3}{2}})+\frac{y^2}{4}=\frac{y^2}{4}+20y^2=\frac{81}{4}y^2,$$

$$x+y^{\frac{3}{2}}+\frac{y}{2}=\pm\frac{9}{2}y;$$

$$\therefore\ x+y^{\frac{3}{2}}-4y=0,\ \text{ and }\ x+y^{\frac{3}{2}}+5y=0.$$

Subtracting the 2nd,

$$y^{\frac{3}{2}}-8=0,\ \text{ and }\ y^{\frac{3}{2}}+9y-8=0;$$

$$\therefore\ y=4,\ \text{ and }\ x=8;\qquad y(y^{\frac{1}{2}}+1)+8(y-1)=0;$$

$$\therefore\ y^{\frac{1}{2}}+1=0,\qquad y+8y^{\frac{1}{2}}-8=0;$$

$$\therefore\ y=1,\qquad y=8(5\mp 2\sqrt{6}),$$

$$x=-4,\qquad x=8(19\mp 8\sqrt{6}).$$

47. $$\left.\begin{array}{r}x^3+y^3+x^2y+xy^2=13\\ x^4y^2+x^2y^4=468\end{array}\right\},\text{ find } x \text{ and } y.$$

Dividing (2) by (1), we have

$$\frac{x^2y^2}{x+y}=36;\quad \therefore\ xy=\pm 6\sqrt{x+y};$$

$$\therefore\ \text{from (1)}\ \{(x+y)^2-12(x+y)^{\frac{1}{2}}\}(x+y)=13,$$

$$\text{or }\ (x+y)^3-12(x+y)^{\frac{3}{2}}=13,$$

whence $x+y=1$, or $(13)^{\frac{2}{3}}$, and $\therefore\ xy=6(13)^{\frac{1}{3}}$, or -6;

$$\therefore\ x-y=\pm 5,\ \text{ or }\ \pm\{-11(13)^{\frac{1}{3}}\}^{\frac{1}{2}};$$

$$\therefore\ x=3,\ \text{or}\ -2,\ \text{or}\ \frac{1}{2}[13^{\frac{2}{3}}\pm\{-11(13)^{\frac{1}{3}}\}^{\frac{1}{2}}],$$

$$y=-2,\ \text{or}\ 3,\ \text{or}\ \frac{1}{2}[13^{\frac{2}{3}}\mp\{-11(13)^{\frac{1}{3}}\}^{\frac{1}{2}}].$$

48. $$\left.\begin{array}{l} x^4+y^4=1+2xy+3x^2y^2 \\ x^3+y^3=(1+x)(1+2y^2) \end{array}\right\}, \text{ find } x \text{ and } y.$$

From the 1st, $\qquad x^2-y^2=1+xy;$

$$\therefore\ x^2-xy+y^2=1+2y^2.$$

Divide the 2nd by this equation,

$$\text{then } x+y=1+x, \text{ and } x^2-xy+y^2=0;$$

$$\therefore\ y=1,\ x^2-x=2;\quad \therefore\ x=2, \text{ or } -1.$$

$$\text{Also}\quad 2y^2+1=0, \text{ and } x^2-xy+\frac{y^2}{4}=-\frac{3y^2}{4};$$

$$\therefore\ y=\pm\frac{1}{\sqrt{-2}};\quad \therefore\ x=\frac{y}{2}(1\pm\sqrt{-3})=\pm\frac{1}{2}\left(\frac{1}{\sqrt{-2}}+\sqrt{\frac{3}{2}}\right).$$

49. $$\left.\begin{array}{l} x^2(b-y)=ay(y-n) \\ y^2(a-x)=bx(x-n) \end{array}\right\}, \text{ find } x \text{ and } y.$$

By subtraction, $\quad n(bx+ay)=xy(x+y).$

By multiplication, &c. $\quad \dfrac{x^2y^2}{ay.bx}=\dfrac{(y-n)(x-n)}{(x-a)(y-b)};$

$$\therefore\ \frac{xy}{ab}=\frac{xy-n(x+y)+n^2}{xy-(bx+ay)+ab}.$$

For $bx+ay$ substitute its equal $\dfrac{xy}{n}(x+y)$; then

$$(xy)^2-\frac{(xy)^2}{n}(x+y)+abxy=abxy-abn(x+y)+abn^2;$$

$$\therefore\ (xy)^2\{n-(x+y)\}=n^2ab\{n-(x+y)\};$$

$$\therefore\ x+y=n, \quad \text{and } xy=\pm n(ab)^{\frac{1}{2}}.$$

$$\text{Whence } x=\frac{1}{2}\{n\pm(n^2\mp 4n\sqrt{ab})^{\frac{1}{2}}\},$$

$$y=\frac{1}{2}\{n\mp(n^2\mp 4n\sqrt{ab})^{\frac{1}{2}}\}.$$

$$\text{Again } b+a\frac{y}{x}=\frac{xy}{n}\left(1+\frac{y}{x}\right)=\pm(ab)^{\frac{1}{2}}\left(1+\frac{y}{x}\right);$$

$$\therefore\ \frac{y}{x}=\frac{\pm(ab)^{\frac{1}{2}}-b}{a\mp(ab)^{\frac{1}{2}}}=\pm\left(\frac{b}{a}\right)^{\frac{1}{2}};$$

$$\therefore\ x^2 = \pm\, n\,(ab)^{\frac{1}{2}} \div \pm\left(\frac{b}{a}\right)^{\frac{1}{2}} = na\,;\quad \therefore\ x = \pm\,(na)^{\frac{1}{2}},$$

$$y^2 = \pm\, n\,(ab)^{\frac{1}{2}} \times \pm\left(\frac{b}{a}\right)^{\frac{1}{2}} = nb\,;\quad \therefore\ y = \pm\,(nb)^{\frac{1}{2}}.$$

56. $$\left.\begin{array}{r} x + (x^2 - y^2)^{\frac{1}{2}} = \dfrac{8}{y}\{(x+y)^{\frac{1}{2}} + (x-y)^{\frac{1}{2}}\} \\ (x+y)^{\frac{3}{2}} - (x-y)^{\frac{3}{2}} = 26 \end{array}\right\}, \text{ find } x \text{ and } y.$$

From (1),

$$x + y + 2\,(x^2 - y^2)^{\frac{1}{2}} + (x - y) = 2 \times \frac{16\,\{(x+y)^{\frac{1}{2}} + (x-y)^{\frac{1}{2}}\}}{(x+y) - (x-y)}$$

$$= \frac{32}{(x+y)^{\frac{1}{2}} - (x-y)^{\frac{1}{2}}};$$

$$\therefore\ (x+y)^{\frac{3}{2}} - (x-y)^{\frac{3}{2}} + (x^2 - y^2)^{\frac{1}{2}}\{(x+y)^{\frac{1}{2}} - (x-y)^{\frac{1}{2}}\} = 32\,;$$

$$\therefore\ (x^2 - y^2)^{\frac{1}{2}}\{(x+y)^{\frac{1}{2}} - (x-y)^{\frac{1}{2}}\} = 32 - 26 = 6\,;$$

$$\therefore\ \frac{(x+y)^{\frac{3}{2}} - (x-y)^{\frac{3}{2}}}{(x+y)^{\frac{1}{2}} - (x-y)^{\frac{1}{2}}}, \text{ or } 2x + (x^2 - y^2)^{\frac{1}{2}} = \frac{26}{6}\,(x^2 - y^2)^{\frac{1}{2}};$$

$$\therefore\ 6 \times 2x = 20\,(x^2 - y^2)^{\frac{1}{2}};$$

$$\therefore\ y^2 = x^2\left(1 - \frac{9}{25}\right);\quad \therefore\ y = \frac{4}{5}x.$$

Hence $x^{\frac{3}{2}}\left\{\left(1 + \frac{4}{5}\right)^{\frac{3}{2}} - \left(1 - \frac{4}{5}\right)^{\frac{3}{2}}\right\}$, or $\left(\frac{x}{5}\right)^{\frac{3}{2}}(9^{\frac{3}{2}} - 1) = 26$;

$$\therefore\ x = 5, \text{ and } y = 4.$$

57. $$\left.\begin{array}{r} (xy^2 + x)^{\frac{1}{2}} + x^{\frac{1}{2}} = y\,(x+9)^{\frac{1}{2}} + 3y \\ x\,(y+1)^2 = 36\left(y^3 + \dfrac{16}{9}\right) \end{array}\right\}, \text{ find } x \text{ and } y.$$

From (1), $$\frac{\sqrt{y^2+1} + 1}{y} = \frac{\sqrt{x+9} + 3}{\sqrt{x}} \ldots\ldots\ldots\ldots(3).$$

Squaring, $$\frac{y^2 + 2 + 2\sqrt{y^2+1}}{y^2} = \frac{x + 18 + 6\sqrt{x+9}}{x};$$

$$\therefore\ \frac{\sqrt{y^2+1} + 1}{y^2} = \frac{3\,(\sqrt{x+9} + 3)}{x}.$$

Dividing (3) by this equation,

$$y = \frac{\sqrt{x}}{3}; \quad \therefore\ x = 9y^2.$$

From (2), $$y^2(y+1)^2 = 4y^3 + \frac{64}{9};$$

$$\therefore\ y(y-1) = \pm\frac{8}{3}.$$

Whence $$y = \frac{1}{6}\{3 \pm \sqrt{105}\}, \text{ or } \frac{1}{6}\{3 \pm \sqrt{-87}\},$$

$$x = \frac{3}{2}\{19 \pm \sqrt{105}\}, \text{ or } \frac{3}{2}\{-13 \pm \sqrt{-87}\}.$$

58. $\left(\frac{x}{a}\right)^{\alpha}\left(\frac{y}{b}\right)^{\beta} = a^{\alpha\beta}, \quad \left(\frac{x}{b}\right)^{\beta}\left(\frac{y}{a}\right)^{\alpha} = b^{\alpha\beta}$, find x and y.

$$\therefore\ x^{\alpha}y^{\beta} = a^{\alpha\beta+\alpha}.\,b^{\beta},$$

$$x^{\beta}y^{\alpha} = a^{\alpha}.\,b^{\alpha\beta+\beta};$$

$$\therefore\ xy = \{a^{\alpha\beta+2\alpha}.\,b^{\alpha\beta+2\beta}\}^{\frac{1}{\alpha+\beta}},$$

$$\frac{x}{y} = \{a^{\alpha\beta}.\,b^{\alpha\beta}\}^{\frac{1}{\alpha-\beta}};$$

$$\therefore\ x^2 = a^{\frac{\alpha\beta+2\alpha}{\alpha+\beta}+\frac{\alpha\beta}{\alpha-\beta}}.\,b^{\frac{\alpha\beta+2\beta}{\alpha+\beta}+\frac{\alpha\beta}{\alpha-\beta}};$$

$$\therefore\ x = \pm\{a^{\alpha(\alpha\beta+\alpha-\beta)}.\,b^{\beta(\alpha^2+\alpha-\beta)}\}^{\frac{1}{\alpha^2-\beta^2}}.$$

Similarly $$y = \pm\{a^{\alpha(\alpha-\beta-\beta^2)}.\,b^{\beta(\alpha-\beta-\alpha\beta)}\}^{\frac{1}{\alpha^2-\beta^2}}.$$

59. $x^{\frac{\sqrt[4]{x}+\sqrt[4]{y}}{2}} = y^{\frac{4}{3}}, \quad y^{\frac{\sqrt[4]{x}+\sqrt[4]{y}}{2}} = x^{\frac{1}{3}}$, find x and y.

Let $$\sqrt[4]{x} + \sqrt[4]{y} = t;$$

then $$x^{\frac{3t}{8}} = y = x^{\frac{2}{3t}};$$

$$\therefore\ \frac{3t}{8} = \frac{2}{3t}; \quad \therefore\ t = \frac{4}{3}.$$

Hence $$x^{\frac{1}{2}} = y, \text{ and } x^{\frac{1}{4}} + x^{\frac{1}{8}} + \frac{1}{4} = \frac{4}{3} + \frac{1}{4};$$

$$\therefore\ x = \left\{-\frac{1}{2} \pm \frac{1}{6}\sqrt{57}\right\}^8, \quad y = \left\{-\frac{1}{2} \pm \frac{1}{6}\sqrt{57}\right\}^4.$$

62. $xyz = 231$, $xyw = 420$, $xzw = 660$, $yzw = 1540$;

find x, y, z, w.

Multiply all together, then

$$(xyzw)^3 = 231 \times 420 \times 660 \times 1540$$
$$= (11 \times 21) \times (6 \times 7 \times 10) \times (6 \times 11 \times 10) \times (2 \times 7 \times 11 \times 10)$$
$$= 11^3 \times 10^3 \times 7^3 \times 6^3;$$
$$\therefore\ xyzw = 6 \times 7 \times 10 \times 11.$$

But $yzw = 1540 = 2 \times 7 \times 10 \times 11$;

$\therefore\ x = 3$. Similarly $y = 7$, $z = 11$, $w = 20$.

63.
$$\left.\begin{array}{l} x^2 + xy + y^2 = 37 \\ x^2 + xz + z^2 = 28 \\ y^2 + yz + z^2 = 19 \end{array}\right\};\ \text{find } x,\ y,\ z.$$

Then $x(y-z) + y^2 - z^2 = 9$; $\therefore\ (x+y+z)(y-z) = 9$.

Again $x^2 - y^2 + z(x-y) = 9$; $\therefore\ (x+y+z)(x-y) = 9$;

$$\therefore\ y - z = x - y;\quad \therefore\ y = \frac{1}{2}(x+z).$$

$$\text{Hence}\quad x^2 + \frac{x}{2}(x+z) + \frac{(x+z)^2}{4} = 37;$$

$$\therefore\ \frac{7x^2 + 4xz + z^2}{x^2 + xz + z^2} = \frac{4 \times 37}{28} = \frac{37}{7};$$

$$\therefore\ 7 + 4t + t^2 = \frac{37}{7}(1 + t + t^2),\ \text{if } t = \frac{z}{x};$$

$$\therefore\ 30t^2 + 9t = 12.$$

$$\text{Hence } t = \frac{1}{2} \text{ or } -\frac{4}{5};$$

$$\therefore\ x^2\left(1 + \frac{1}{2} + \frac{1}{4}\right) = 28;$$

$$\therefore\ x = \pm 4,\ z = \pm 2,\ y = \pm 3.$$

$$\text{Also } x^2\left(1 - \frac{4}{5} + \frac{16}{25}\right) = 28;$$

$$\therefore\ x = \pm \frac{10}{3}\sqrt{3},\quad z = \mp \frac{8}{3}\sqrt{3},\quad y = \pm \frac{1}{3}\sqrt{3}.$$

64. $$\left.\begin{aligned} x+y+z&=11\ldots\ldots\ldots(1)\\ x^2+y^2+z^2&=49\ldots\ldots\ldots(2)\\ yz&=3x(z-y)\ \ldots\ldots\ldots(3)\end{aligned}\right\};\quad \text{find } x, y, z.$$

Subtract (2) from the square of (1) and divide by 2,

$$\text{then}\quad xy+xz+yz=\frac{121-49}{2}=36.$$

Hence, and from (3),

$$\frac{36-xz}{x+z}=y=\frac{3xz}{3x+z}.$$

Substituting these values of y successively in (1), we get

$$3x^2+7xz+z^2=11(3x+z)\ldots\ldots\ldots\ldots\ldots(4),$$

$$x^2+xz+z^2=11(x+z)-36\ \ldots\ldots\ldots(5).$$

Multiply (5) by 3 and subtract from (4),

$$\text{hence}\quad 4xz-2z^2=-22z+108;$$

$$\therefore\ x=\frac{z^2-11z+54}{2z},$$

$$x+z=\frac{3z^2-11z+54}{2z},\qquad xz=\frac{z^2-11z+54}{2}.$$

Substituting in $(x+z)^2-11(x+z)=xz-36$,

$$\left(\frac{3z^2-11z+54}{2z}\right)\left(\frac{3z^2-33z+54}{2z}\right)=\frac{z^2-11z-18}{2};$$

$$\therefore\ \frac{3(3z^2-11z+54)}{2z^2}=\frac{z^2-11z-18}{z^2-11z+18}.$$

Subtracting both sides from 4, we get

$$-\frac{z^2+33z-162}{2z^2}=\frac{3z^2-33z+90}{z^2-11z+18};$$

$$\therefore\ -\frac{(z-6)(z-27)}{2z^2}=\frac{3(z-5)(z-6)}{(z-2)(z-9)}.$$

$$\text{Whence}\quad z-6=0;\quad \therefore\ z=6;$$

$$\therefore\ x=\frac{54-6\times 5}{2\times 6}=2;\quad \therefore\ y=3.$$

65. $$\left.\begin{aligned} x+y+z&=13\\ x^2+y^2+z^2&=91\\ y^2&=xz \end{aligned}\right\};\ \text{find } x,\ y, \text{ and } z.$$

Subtract the 2nd from the square of the 1st, and divide by 2,

$$\text{then } xy+xz+yz=\frac{169-91}{2}=39.$$

For xz write y^2,

$$\text{then } y(x+y+z)=39;$$

$$\therefore y=3;$$

$$\therefore x+z=10, \text{ and } xz=9;$$

$$\therefore x=9, \text{ or } 1, \text{ and } z=1, \text{ or } 9.$$

66. $$\left.\begin{aligned} x+y+z&=13\\ x^2+y^2+z^2&=61\\ x(y+z)&=2yz \end{aligned}\right\};\ \text{find } x,\ y, \text{ and } z.$$

Subtract the 2nd from the square of the 1st, and divide by 2,

$$\text{then } xy+xz+yz=\frac{13^2-61}{2}=54.$$

From the 3rd, $xy+xz+yz=3yz$;

$$\therefore yz=18.$$

Multiply the 1st by x, subtract the 3rd and substitute for yz,

$$\text{then } x^2=13x-36.$$

Whence $x=4$, or 9;

$\therefore y+z=9$, taking the former value of x,

$yz=18$;

$\therefore y=3$, or 6, and $z=6$, or 3.

Ex. 43.

16. Let $x+y=$ the hypothenuse,

and $x-y=$ one side.

Then by the question, $4xy=2$ .... (1),

and $4\{(x+y)^2+(x-y)^2\}=5\{(x+y)+(x-y)\}$ .... (2),

from (1), $(4xy)^{\frac{1}{2}} = \sqrt{2}$ the remaining side,

from (2), $4(x^2 + y^2) = 5x$ .... (3),

from (1) and (3), $4\left(x^2 + \dfrac{1}{4x^2}\right) = 5x$,

whence $4x^4 - 5x^3 + 1 = 0$, or $4x^3(x-1) - (x^3 - 1) = 0$;

$$\therefore\ x - 1 = 0, \quad x = 1;$$

$$\therefore\ y = \frac{1}{2};$$

$$\therefore\ x + y = \frac{3}{2}, \text{ the hypothenuse,}$$

$$x - y = \frac{1}{2}, \text{ one side,}$$

$$(4xy)^{\frac{1}{2}} = \sqrt{2}, \text{ the other side.}$$

18 Let x, y, z be the numbers.

Then $x + y + z = 26$ .... (1),

$x^2 - y^2 = y^2 - z^2$ .... (2),

$x^2 + y^2 + z^2 = 300$ .... (3).

From (2), $x^2 - 2y^2 + z^2 = 0$;

$$\therefore\ 3y^2 = 300; \quad \therefore\ y = 10,$$

$$x^2 + z^2 = 200,$$

$$x + z = 16;$$

$$\therefore\ 2xz = 56,$$

$$x - z = \sqrt{200 - 56} = 12;$$

$$\therefore\ x = \frac{1}{2}(16 + 12) = 14, \quad z = 2.$$

19. Let x, y, z, w be the numbers.

Then $x + y + z + w = 44$.............. (1),

$xy + zw = 250$ .... (2),

$xz + yw = 234$ .... (3),

$xw + yz = 225$ .... (4).

From (2) and (3), $(x+w)(y+z)=484.$

From (1), $y+z=44-(x+w)$;

$$\therefore (x+w)^2-44(x+w)+484=0;$$

$$\therefore x+w=22.$$

From (2) and (4), $(x+z)(y+w)=475.$

Then, as before, $x+z=25.$

From (3) and (4) $(x+y)(z+w)=459,$

hence $x+y=27$;

$$\therefore (x+w)+(x+z)+(x+y)=22+25+27=74;$$

$$\therefore 2x=74-44=30;$$

$$\therefore x=15, \quad y=12, \quad z=10, \quad w=7.$$

INEQUALITIES.

Ex. 44.

1. $$n^3+1>n^2+n, \text{ if } n^3+1>n(n+1),$$

$$\text{or if } (n^2-n+1)>n, \text{ or if } n^2+1>2n,$$

which is the case since n^2-2n+1 or $(n-1)^2$ is essentially positive whatever be the value of n;

$$\therefore n^2+1>2n;$$

$$\therefore n^3+1>n^2+n.$$

4. $$\frac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3>y^3+y^3+y^3+y^3,$$

$$\text{and } <x^3+x^3+x^3+x^3,$$

$$\text{i.e. } \frac{x^4-y^4}{x-y}<4x^3, \text{ but } >4y^3;$$

$$\therefore x-y>\frac{x^4-y^4}{4x^3}, \text{ and } <\frac{x^4-y^4}{4y^3}.$$

9. Show that $(a+b+c)^3>27abc$, but $<9(a^3+b^3+c^3)$.

$$\text{Now } ab<\left(\frac{a+b}{2}\right)^2, \quad cd<\left(\frac{c+d}{2}\right)^2;$$

$$\therefore abcd<\left(\frac{a+b}{2}\cdot\frac{c+d}{2}\right)^2.$$

$$\text{But } \frac{a+b}{2}\cdot\frac{c+d}{2} < \left\{\frac{\frac{1}{2}(a+b)+\frac{1}{2}(c+d)}{2}\right\}^2;$$

$$\therefore\ abcd < \left(\frac{a+b+c+d}{4}\right)^4.$$

As the value of d is arbitrary, assume $d = \dfrac{a+b+c}{3}$.

$$\text{Then } abcd < \left(\frac{3d+d}{4}\right)^4 < d^4,$$

$$abc < d^3 < \left(\frac{a+b+c}{3}\right)^3;$$

$$\therefore\ (a+b+c)^3 > 27abc.$$

$$\text{Again } ab < \left(\frac{a+b}{2}\right)^2;$$

$$\therefore\ ab(a+b) < \frac{(a+b)^3}{4};$$

$$\therefore\ a^3+b^3+3ab(a+b) \text{ or } (a+b)^3 < a^3+b^3+\frac{3}{4}(a+b)^3;$$

$$\therefore\ \frac{a^3+b^3}{2} > \left(\frac{a+b}{2}\right)^3.$$

$$\text{Also } \frac{c^3+h^3}{2} > \left(\frac{c+h}{2}\right)^3;$$

$$\therefore\ \frac{a^3+b^3+c^3+h^3}{4} > \left(\frac{a+b+c+h}{4}\right)^3.$$

$$\text{Let } h = \frac{a+b+c}{3},$$

$$\text{then } \frac{a^3+b^3+c^3+h^3}{4} > \left(\frac{3h+h}{4}\right)^3 > h^3;$$

$$\therefore\ a^3+b^3+c^3 > 3h^3 > 3\left(\frac{a+b+c}{3}\right)^3;$$

$$\therefore\ (a+b+c)^3 < 9(a^3+b^3+c^3).$$

11. $x^2+y^2 > 2xy;\quad \therefore\ x^3+xy^2 > 2x^2y,$ and $x^2y+y^3 > 2xy^2;$

$$\therefore\ x^3+y^3+x^2y+xy^2 > 2x^2y+2xy^2,$$

$$\text{or } x^3+y^3 > x^2y+xy^2.$$

Similarly $x^3+z^3>x^2z+xz^2$,

and $y^3+z^3>y^2z+yz^2$;

$\therefore$ by addition

$$2(x^3+y^3+z^3)>x^2y+xy^2+x^2z+xz^2+y^2z+yz^2,$$

or $x^3+y^3+z^3>\frac{1}{2}\{x^2y+xy^2+x^2z+xz^2+y^2z+yz^2\}$.

Ex. 45.

10. Since $a:b=c:d$; $\therefore a-b:a=c-d:c$.........(1),

and $a^n-b^n:a^n=c^n-d^n:c^n$ ....(2).

Now from (1), $\dfrac{a-b}{a}=\dfrac{c-d}{c}$,

and from (2), $\dfrac{a^n-b^n}{a^n}=\dfrac{c^n-d^n}{c^n}$,

but by hypothesis $a>c$; and $\therefore a^n>c^n$;

$\therefore a-b>c-d$, and $a^n-b^n>c^n-d^n$,

and $\therefore a+d>b+c$, and $a^n+d^n>b^n+c^n$.

11. (1) $a-c-b+d=a-c-b+\dfrac{bc}{a}$

$$=\frac{a(a-c)-b(a-c)}{a}=\frac{(a-b)(a-c)}{a}.$$

(2) $\left(\frac{1}{a}+\frac{1}{d}\right)-\left(\frac{1}{b}+\frac{1}{c}\right)=\left(\frac{1}{a}+\frac{a}{bc}\right)-\left(\frac{1}{b}+\frac{1}{c}\right)$

$$=\frac{bc+a^2-ac-ab}{abc}$$

$$=\frac{a(a-c)-b(a-c)}{abc}=\frac{(a-b)(a-c)}{abc};$$

$$\therefore \frac{1}{a}+\frac{1}{d}=\frac{1}{b}+\frac{1}{c}+\frac{(a-b)(a-c)}{abc}.$$

12. Since $y\propto x$; $\therefore \dfrac{y}{x}=\dfrac{21}{3}=7$;

$\therefore y=7x$ is the equation required.

18. Let the value of diamonds $= r \times$ square of their weight,

and .... rubies $= s \times (\text{weight})^{\frac{3}{2}}$,

$$\text{then (1)},\quad ra^2 = smb^{\frac{3}{2}},$$

$$\text{and}\quad ra^2 + smb^{\frac{3}{2}} = c,$$

$$\text{also}\quad ra^2 + sb^{\frac{3}{2}} = c;$$

$$\therefore\ s = \frac{c}{(m+1)\,b^{\frac{3}{2}}}.$$

$$\text{Again,}\quad mra^2 + ra^2 = mc;$$

$$\therefore\ r = \frac{mc}{(m+1)\,a^2};$$

$$\therefore\ \text{value of the diamond} = rx^2 = \frac{mcx^2}{(m+1)\,a^2}\,\pounds.$$

$$\text{and value of the ruby} = sx^{\frac{3}{2}} = \frac{cx^{\frac{3}{2}}}{(m+1)\,b^{\frac{3}{2}}}\,\pounds.$$

Arithmetical Progression.

Ex. 46.

42. Now $\dfrac{S_n}{S'_n} = \dfrac{2a + (n-1)d}{2a' + (n-1)d'} = \dfrac{13 - 7n}{1 + 3n}$.

Let $n = 1$; then $\dfrac{a}{a'} = \dfrac{13-7}{1+3} = \dfrac{3}{2}$, the ratio of the first terms,

$n = 3$; then $\dfrac{a+d}{a'+d'} = \dfrac{13-21}{1+9} = -\dfrac{4}{5}$,

the ratio of the second terms.

49. In the 1st series, the p^{th} term $= 2 + (p-1)\,3 = 3p - 1$,

...... 2nd .... q^{th} ...... $= 3 + (q-1)\,4 = 4q - 1$.

If these terms be identical, $3p = 4q$; $\therefore\ \dfrac{p}{q} = \dfrac{4}{3}$.

Hence, every 4th term of the 1st series is identical with every 3rd term of the 2nd series; there are, therefore, 25 terms identical, or common to the two series.

Ex. 47.

1. $$2^3-1^3=(1+1)^3-1^3=3\times 1^2+3\times 1+1,$$
$$3^3-2^3=(2+1)^3-2^3=3\times 2^2+3\times 2+1,$$
$$4^3-3^3=(3+1)^3-3^3=3\times 3^2+3\times 3+1,$$
$$\&c.=\&c.,$$
$$(n+1)^3-n^3= \qquad 3n^2+3n+1.$$

Adding the vertical columns we have

$$(n+1)^3-n^3=3(1^2+2^2+3^2+\ldots+n^2)+3(1+2+3+\ldots+n)+n;$$

$$\therefore\ 1^2+2^2+3^2+\ldots\ldots+n^2=\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}=\frac{1}{6}n(n+1)(2n+1).$$

And if $n=15$, we have

$$1^2+2^2+3^2+\&c. \text{ to } 15 \text{ terms}=\frac{1}{6}.15.16.31,$$

or the sum $=1240$.

8. Given series $=2^2+6-(4^2+12)+(6^2+18)-\&c.;$

$\therefore$ the sum $=2^2-4^2+6^2-\&c.$ to $2r$ terms

$$+6(1-2+3-4+\&c. \text{ to } 2r \text{ terms})$$
$$=4\{(1-4)+(9-16)+(25-36)+\&c. \text{ to } 2r \text{ terms}\}$$
$$+6\{(1-2)+(3-4)+\&c. \text{ to } 2r \text{ terms}\}$$
$$=-4\{3+7+11+15+\&c. \text{ to } r \text{ terms}\}$$
$$-6\{1+1+1+\&c. \text{ to } r \text{ terms}\}$$
$$=-(8r^2+4r)-6r$$
$$=-2r(4r+5).$$

Geometrical Progression.

Ex. 48.

1. Let z = the 5^{th} term, a the 1^{st} term, r = the common ratio,

$$\text{then } z = ar^{n-1} = 5 \times 2^4 = 5 \times 16 = 80.$$

5. $$\text{Here } a = -21, \quad r = -\frac{2}{3}, \quad n = 7;$$

$$\therefore \text{ the } 7^{\text{th}} \text{ term} = ar^{n-1} = -21 \times \left(-\frac{2}{3}\right)^6 = -\frac{448}{243}.$$

11. $$\text{Here } a = \frac{1}{3}, \quad r = -\frac{3}{2}, \quad n = 8;$$

$$\therefore S = \frac{1}{3} \cdot \frac{\left(-\frac{3}{2}\right)^8 - 1}{-\frac{3}{2} - 1} = \frac{6561 - 256}{-384 \times 5} = -\frac{1261}{384}.$$

15. $$\text{Here } a = \frac{2}{5}, \quad r = -\left(\frac{5}{2}\right)^{\frac{1}{2}};$$

$$\therefore S = \frac{2}{5} \cdot \frac{\left\{-\left(\frac{5}{2}\right)^{\frac{1}{2}}\right\}^n - 1}{-\left(\frac{5}{2}\right)^{\frac{1}{2}} - 1} = \frac{1 - \left\{-\left(\frac{5}{2}\right)^{\frac{1}{2}}\right\}^n}{1 + \left(\frac{5}{2}\right)^{\frac{1}{2}}} \cdot \frac{2}{5}$$

$$= \{1 - (-\sqrt{2})^n\} \cdot \frac{2^{\frac{3}{2}}}{5(\sqrt{5} + \sqrt{2})}$$

$$= \{1 - (-\sqrt{2})^n\} \cdot \frac{2^{\frac{3}{2}}}{15}(\sqrt{5} - \sqrt{2}).$$

19. $$\text{Here } a = \frac{2}{3}, \quad r = -\frac{3}{4};$$

$$\therefore S = \frac{a}{1 - r} = \frac{2}{3} \div \left(1 + \frac{3}{4}\right) = \frac{2}{3} \times \frac{4}{7} = \frac{8}{21}.$$

28. $$S - 1 = \frac{f-g}{g} \div \left(1 + \frac{x}{g}\right) = \frac{f-g}{g+x};$$

$$\therefore\ S = \frac{f+x}{g+x}.$$

32. Find the value of $\cdot 02\dot{7}$.

$$S = \frac{2}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \ldots = \frac{1}{50} + \frac{7}{10^3} \cdot \frac{1}{1 - \frac{1}{10}} = \frac{1}{50} + \frac{7}{900} = \frac{1}{36}.$$

Otherwise, $S = \cdot 02\dot{7}$;

$$\therefore\ 1000S = 27\cdot777\ldots\ldots$$
$$100S = 2\cdot777\ldots\ldots$$
$$\therefore\ S = \frac{25}{900} = \frac{1}{36}.$$

37. Let a be the first term, and r the common ratio of the series, then

the p^{th} term $= ar^{p-1} = P$...............(1),

and the q^{th} term $= ar^{q-1} = Q$...............(2),

the n^{th} term $= ar^{n-1}$,

from (1) and (2) we have

$$r^{p-q} = \frac{P}{Q}, \text{ whence } r = \left(\frac{P}{Q}\right)^{\frac{1}{p-q}};$$

$$\therefore\ ar^{n-1} = ar^{p-1} \cdot r^{n-p} = P\left(\frac{P}{Q}\right)^{\frac{n-p}{p-q}} = \left(\frac{P^{n-q}}{Q^{n-p}}\right)^{\frac{1}{p-q}}.$$

40. $(a+b+c+d)^2 = (a+b)^2 + (c+d)^2 + 2(a+b)(c+d)$.

If r be the common ratio, then

$$(a+b)(c+d) = a(1+r)\, ar^2(1+r) = (ar+ar^2)^2 = (b+c)^2;$$

$$\therefore\ (a+b+c+d)^2 = (a+b)^2 + (c+d)^2 + 2(b+c)^2.$$

41. Let r be the common ratio,

then $a^2 + b^2 + c^2 >$ or $< (a-b+c)^2$,

according as $1 + r^2 + r^4 >$ or $< (1 - r + r^2)^2$,

.... $1 + r^2 + r^4 >$ or $< (1 + r^2 + r^4 - 2r + 2r^2 - 2r^3$,

.... $2r(1 - 2r + r^2) >$ or $< -2r^2$.

If r be positive, then $(1-r)^2 > -r$,

.... negative, ... $(1+r)^2 > +r$.

Hence $a^2+b^2+c^2$ is $> (a-b+c)^2$.

44. Let a, ar, ar^2, &c. to $2n$ terms be the series, then

$$ar + ar^3 + ar^5 + \&c. \text{ to } n \text{ terms} = ar \cdot \frac{r^{2n}-1}{r^2-1} = S_1 \ldots\ldots (1),$$

$$\text{and } a + ar^2 + ar^4 + \&c. \text{ to } n \text{ terms} = a \cdot \frac{r^{2n}-1}{r^2-1} = S_2 \ldots\ldots (2).$$

Dividing (1) by (2), we have

$$r = \frac{S_1}{S_2},$$

which, substituted in (1), gives

$$a = \frac{S^{2n}}{S_2} \cdot \frac{S_1^2 - S_2}{S_1^{2n} - S_2^{2n}}.$$

The p^{th} and q^{th} terms of the series are respectively

$$ar^{p-1} \text{ and } ar^{q-1},$$

and by the question, the p^{th} and q^{th} terms are the first and last terms respectively of an A.R. series of $(m+2)$ terms;

$$\therefore \text{ the common difference } = \frac{a(r^{q-1} - r^{p-1})}{m+1};$$

$$\therefore \text{ the } r^{\text{th}} \text{ mean} = \text{the } (r+1)^{\text{th}} \text{ term } = \frac{ar^{p-1} + ar(r^{q-1} - r^{p-1})}{m+1}$$

$$= a\left\{r^{p-1}\left(1 - \frac{r}{m+1}\right) + \frac{r}{m+1} \cdot r^{q-1}\right\}$$

$$= a\left\{\left(\frac{m-r+1}{m+1}\right) \cdot r^{p-1} + \frac{r}{m+1} \cdot r^{q-1}\right\}$$

$$= \frac{a}{r}\left\{\left(\frac{m-r+1}{m+1}\right) r^p + \frac{r}{m+1}\, r^q\right\}$$

$$= \frac{S_2^{2n}}{S_1}\left\{\frac{r}{m+1}\left(\frac{S_1}{S_2}\right)^q + \frac{m-r+1}{m+1}\left(\frac{S_1}{S_2}\right)^p\right\} \cdot \frac{S_1^2 - S_2^2}{S_1^{2n} - S_2^{2n}}.$$

Ex. 49.

1. $$\text{Let } S = 1 - 3x + 5x^2 - 7x^3 + \&c.;$$
$$\therefore\ xS = \quad x - 3x^2 + 5x^3 - \&c.;$$
$$(1+x)S = 1 - 2x + 2x^2 - 2x^3 + \&c.$$
$$= 1 - \frac{2x}{1+x} = \frac{1-x}{1+x};$$
$$\therefore\ S = \frac{1-x}{(1+x)^2}.$$

8. Let $S = 1.1 + 2.3 + 4.5 + 8.7 + \ldots + (2n-1).2^{n-1};$
$$\therefore\ 2S = \quad 2.1 + 4.3 + 8.5 + \ldots + (2n-3)2^{n-1} + (2n-1)2^n;$$
$$\therefore\ -S = 1 + 4 + 8 + 16 + \&c. + 2^n - (2n-1)2^n$$
$$= 1 - (2n+1)2^n + 4.\frac{2^n - 1}{2-1}$$
$$= -2^n(2n-3) - 3;$$
$$\therefore\ S = 2^n(2n-3) + 3.$$

9. Let $S = 1.2x + 2.3x^2 + 3.4x^3 + 4.5x^4 + \ldots\ldots$ to infinity,

then $xS = \quad + 1.2x^2 + 2.3x^3 + 3.4x^4 + \ldots\ldots;$
$$\therefore\ (1-x)S = 1.2x + 2.2x + 3.2x^2 + 4.2x^4 + \ldots\ldots$$
$$= 2x(1 + 2x + 3x^2 + 4x^3 + \ldots\ldots).$$

Now, let $R = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots,$

then $xR = \quad x + 2x^2 + 3x^3 + \ldots\ldots;$
$$\therefore\ (1-x)R = 1 + x + x^2 + x^3 + \ldots\ldots = \frac{1}{1-x};$$
$$\therefore\ R = \frac{1}{(1-x)^2};$$
$$\therefore\ S = \frac{2x}{(1-x)^3}.$$

Similarly, for the sum of n terms.

Harmonical Progression.

Ex. 50.

7. Let a be the 1st term and d the common difference of an A.P. series of which the m^{th} and n^{th} terms are $\frac{1}{M}$ and $\frac{1}{N}$ respectively, then

$$\text{the } m^{th} \text{ term} = a + (m-1)\,d = \frac{1}{M},$$

$$\text{the } n^{th} \text{ term} = a + (n-1)\,d = \frac{1}{N},$$

$$\text{whence } (m-n)\,d = \frac{N-M}{MN}; \; \therefore\; d = \frac{N-M}{MN(m-n)}.$$

The $(m+n)^{th}$ term $= a + (m+n-1)\,b = a + (m-1)\,b + nb$

$$= \frac{1}{M} + \frac{n(N-M)}{NM(m-n)} = \frac{mN-nM}{(m-n)\,NM};$$

$\therefore$ the $(m+n)^{th}$ term of the Harm. Progression is

$$\frac{(m-n)\,NM}{mN-nM}.$$

9. Since a, b, c are in Harm. Progression,

$$b = \frac{2ac}{a+c}.$$

$$\text{Now } a = \left(\frac{a+c}{2} + \frac{a-c}{2}\right),$$

$$c = \left(\frac{a+c}{2} - \frac{a-c}{2}\right);$$

$$\therefore\; a^2 + c^2 = 2\left\{\left(\frac{a+c}{2}\right)^2 + \left(\frac{a-c}{2}\right)^2\right\}.$$

$$\text{Hence } \frac{a^2+c^2}{2} > \left(\frac{a+c}{2}\right)^2 > \left(\frac{2ac}{a+c}\right)^2;$$

$$\therefore\; a^2 + c^2 > 2b^2.$$

Again, $a^n + c^n = \left(\frac{a+c}{2} + \frac{a-c}{2}\right)^n + \left(\frac{a+c}{2} - \frac{a-c}{2}\right)^n$

$$= 2\left\{\left(\frac{a+c}{2}\right)^n + \frac{n(n-1)}{1.2}\left(\frac{a+c}{2}\right)^{n-2}\left(\frac{a-c}{2}\right)^2 + \ldots\right\},$$

since all the terms of the series are positive, n being a positive integer;

$$\therefore\ a^n + c^n > 2\left(\frac{a+c}{2}\right)^n > 2\left(\frac{2ac}{a+c}\right)^n > 2b^n.$$

17. Let r be the common ratio of Geom. Progression.

Then $a^x = (ar)^y = (ar^2)^z = \&c.$;

$$\therefore a^{\frac{x-y}{y}} = r = a^{\frac{x-z}{2z}};$$

$$\therefore \frac{x-y}{y} = \frac{x-z}{2z};$$

$$\therefore y = \frac{2xz}{x+z}.$$

Hence x, y, z are in Harm. Progression.

PILES OF BALLS AND SHELLS.

Ex. 51.

9. Number of balls

$$= \frac{1}{6}n(n+1)(2n+1+3d) - \frac{1}{6}m(m+1)(2m+1+3d).$$

Here $n = 25 + 34 = 59$, and $m = 34$;

$$d = 100 - 59 = 41 = 75 - 34;$$

$\therefore$ required number $= \frac{1}{6}(59 \times 60 \times 242 - 34 \times 35 \times 192) = 104700.$

14. $\frac{1}{6}n(n+1)(n+2) : \frac{1}{6}n(n+1)(2n+1) = 6 : 11$;

$\therefore\ n+2 : 2n+1 = 6 : 11$, whence $n = 16$;

$$\therefore \frac{1}{6}n(n+1)(n+2) = \frac{1}{6}.16.17.18 = 816,$$

and $\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}.16.17.23 = 1496.$

PERMUTATIONS AND COMBINATIONS.

Ex. 52.

4. The whole number of letters is 11; and a recurs 5 times, and e, 4 times;

$\therefore$ required number

$$=\frac{1.2.3.4.5.6.7.8.9.10.11}{1.2.3.4\times 1.2.3.4}=7.2.9.10.11$$

$$=13860.$$

10. $p_3-p_2=n(n-1)\{(n-2)-1\}=n(n-1)(n-3)=\frac{p_4}{n-2}$,

$$p_4-p_3=n(n-1)(n-2)\{(n-3)-1\}$$

$$=n(n-1)(n-2)(n-4)=\frac{p_5}{n-3},$$

&c. = &c.

$$p_{n-1}-p_{n-2}=\frac{p_n}{2};$$

$$\therefore (p_3-p_2)(p_4-p_3)\dots(p_{n-1}-p_{n-2})$$

$$=\frac{p_4 \cdot p_5 \cdots p_n}{(n-2)(n-3)\dots 2}=\frac{p_2 p_3 \cdot p_4 \cdot p_5 \cdots p_n}{p_3\times \underline{|n}}=\frac{p_2 p_3 \cdots p_n}{p_3 p_n};$$

$$\therefore p_2 p_3 \cdots p_n = p_3 p_n\{(p_3-p_2)(p_4-p_3)\dots(p_{n-1}-p_{n-2})\}.$$

14. (1) $C_5=\frac{36\cdot 35\cdot 34\cdot 33\cdot 32}{1.2.3.4.5}=376992.$

(2) Remove the given patrol from the 36, then

$$C_4=\frac{35\cdot 34\cdot 33\cdot 32}{1.2.3.4}=52360.$$

20. If $C_m^{(r)}$ denote the number of combinations of m things taken r at a time,

Then the total number

$$C_{2n}=C_{2n}^{(1)}+C_{2n}^{(2)}+C_{2n}^{(3)}+\dots+C_{2n}^{(2n-1)}+C_{2n}^{(2n)}$$

$$=2n+\frac{2n(2n-1)}{1.2}+\frac{2n(2n-1)(2n-2)}{1.2.3}+\dots+2n+1$$

$$=(1+1)^{2n}-1=2^{2n}-1.$$

$$\text{Similarly,}\quad C_n = 2^n - 1.$$

$$\text{Hence}\quad 2^{2n} - 1 = 65\,(2^n - 1);$$

$$\therefore\ 2^n + 1 = 65,$$

$$2^n = 64 = 2^6;$$

$$\therefore\ n = 6.$$

23. Combinations of consonants, 4 together

$$= \frac{19 \cdot 18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4},$$

combinations of vowels, 2 together $= \dfrac{5 \cdot 4}{1 \cdot 2}$;

$\therefore$ number of different sets, each having 4 consonants and 2 vowels $= \dfrac{19 \cdot 18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4} \times \dfrac{5 \cdot 4}{1 \cdot 2}$;

$$\therefore \text{ number of words} = \frac{19 \cdot 18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{5 \cdot 4}{1 \cdot 2} \times 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$$

$$= 27907200.$$

26.
$$C_n^{(r-1)} = \frac{n\,(n-1)\,(n-2)\ldots(n-r+2)}{1 \cdot 2 \cdot 3 \ldots (r-1)},$$

$$\text{and}\quad C_n^{(r)} = \frac{n-r+1}{r} \times C_n^{(r-1)};$$

$$\therefore\ C_n^{(r)} + C_n^{(r-1)} = C_n^{(r-1)} \times \frac{n+1}{r}.$$

$$\text{But}\quad C_{n+1}^{(r)} = \frac{(n+1)\,n\,(n-1)\,(n-2)\ldots(n-r+2)}{1 \cdot 2 \cdot 3 \ldots r}$$

$$= \frac{n\,(n-1)\,(n-2)\ldots(n-r+2)}{1 \cdot 2 \cdot 3 \ldots (r-1)} \times \frac{n+1}{r}$$

$$= C_n^{(r-1)} \times \frac{n+1}{r}$$

$$= C_n^{(r)} + C_n^{(r-1)}.$$

28.
$$C_{2n}^{(r)} = \frac{2n\,(2n-1)\ldots(2n-r+1)}{1 \cdot 2 \ldots r},$$

$$C_{2n}^{(r-1)} = \frac{2n\,(2n-1)\ldots(2n-r+2)}{1 \cdot 2 \ldots (r-1)};\quad C_{2n}^{(r+1)} = \frac{2n\,(2n-1)\ldots(2n-r)}{1 \cdot 2 \ldots (r+1)}.$$

Now $C_{2n}^{(r)} > C_{2n}^{(r-1)}$, and also $> C_{2n}^{(r+1)}$,

$$\frac{2n-r+1}{r} > 1, \text{ and } \frac{2n-r}{r+1} < 1;$$

$$\therefore r < n + \frac{1}{2}, \text{ and } > n - \frac{1}{2};$$

$$\therefore r = n.$$

30. The number of combinations of the p things taken $1, 2, 3 \ldots p$ together $= p$,

the number of combinations of the q things taken $1, 2, 3 \ldots q$ together $= q$;

therefore the number of combinations of the $p+q$ things taken $1, 2, 3 \ldots (p+q)$ together $= pq + p + q$

$$= (p+1)(q+1) - 1.$$

Hence the whole number of combinations of $p+q+r$ things, p being of one sort, q of another sort, and r of a third, taken $1, 2, 3 \ldots (p+q+r)$ together

$$= (pq+p+q)r + (pq+p+q) + r = (pq+p+q+1)(r+1) - 1$$
$$= (p+1)(q+1)(r+1) - 1,$$

and so on for any additional sets of like things;

therefore the required number $= (p+1)(q+1)(r+1) \ldots - 1$,

where $p + q + r + \ldots = n$.

Binomial and Multinomial Theorems.

Ex. 53.

4. $$(3a^{-1} - 2x)^{-4} = \left(\frac{a}{3}\right)^4 \left\{1 - \frac{2}{3}ax\right\}^{-4} = \frac{a^4}{81}\left\{1 + 4\left(\frac{2ax}{3}\right)\right.$$

$$\left. + \frac{4 \cdot 5}{1 \cdot 2}\left(\frac{2ax}{3}\right)^2 + \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3}\left(\frac{2ax}{3}\right)^3 + \frac{4 \cdot 5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4}\left(\frac{2ax}{3}\right)^4 + \&c.\right\}$$

$$= \frac{a^4}{81} + \frac{8a^5x}{243} + \frac{40a^6x^2}{729} + \frac{160a^7x^3}{2187} + \frac{560a^8x^4}{6561} + \&c.$$

6_4. $$(x^5+z^5)^{-\frac{2}{5}} = x^{-2}\left\{1 - \frac{2}{5}\cdot\frac{z^5}{x^5} + \frac{2}{5}\cdot\frac{\left(\frac{2}{5}+1\right)}{2}\cdot\frac{z^{10}}{x^{10}}\right.$$

$$\left. - \frac{2}{5}\cdot\frac{\frac{2}{5}+1}{2}\cdot\frac{\frac{2}{5}+2}{3}\cdot\frac{z^{15}}{x^{15}} + \&c. \pm \frac{2}{5}\cdot\frac{\frac{2}{5}+1}{2}\cdot\&c.\,\frac{\frac{2}{5}+r-1}{r}\cdot\frac{z^{5r}}{x^{5r}} \mp \&c.\right\}$$

$$= x^{-2} - \frac{2}{5}x^{-7}z^5 + \frac{7}{25}x^{-12}z^{10} - \frac{28}{125}x^{-17}z^{15} + \&c.$$

$$\pm \frac{2\cdot 7\cdot 12\cdot \&c.\ldots(5r-3)}{1\cdot 2\cdot 3\cdot \&c.\, r\cdot 5^r}\cdot x^{-(5r+6)}z^{5r} \mp \&c.$$

9_1. $$(1+x+x^2)^5 = (1+x)^5+5(1+x)^4.x^2+10(1+x)^3.x^4+10(1+x)^2x^6 +5(1+x)x^8+x^{10}$$

$$= 1+5x+10x^2+10x^3+5x^4+x^5+5x^2+20x^3+30x^4 +20x^5+5x^6+10x^4+30x^5+30x^6+10x^7+10x^6 +20x^7+10x^8+5x^8+5x^9+x^{10}$$

$$= 1+5x+15x^2+30x^3+45x^4+51x^5+45x^6+30x^7 +15x^8+5x^9+x^{10}.$$

10_1. $$(1+x+x^2+\&c.)^2 = \left(\frac{1}{1-x}\right)^2 = (1-x)^{-2}$$

$$= 1+2x+3x^2+4x^3+\&c.$$

14. $$(1+3x-x^2)^5 = \{1+(3-x)\,x\}^5.$$

The terms involving x^4 are

$$\frac{5\cdot 4}{1\cdot 2}(3-x)^2x^2,\quad \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}(3-x)^3x^3,\quad \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}(3-x)^4x^4.$$

Hence coefficient of

$$x^4 = 10 - 10\times 27 + 5\times 81 = 415 - 270 = +145.$$

17. In the expression

$$\frac{n(n-1)(n-2)\,\&c.\,(n-r+1)}{1\cdot 2\cdot 3\ldots\ldots r}a^{n-r}x^r;$$

for r write 5, and for x put $(b+c)$, and we have

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}a^{n-5}(b+c)^5.$$

Now the term in $(b+c)^5$ involving b^2c^3 is $10b^2c^3$; therefore the required coefficient

$$=\frac{n(n-1)(n-2)(n-3)(n-4)}{12\times 10}\times 10$$

$$=\frac{n(n-1)(n-2)(n-3)(n-4)}{12}.$$

22. $(1+2x+3x^2+\&c.)^2=(1-x)^{-4}$

$$=1+4x+4\cdot\frac{4+1}{2}x^2+\&c.+\frac{4(4+1)(4+2).\&c.(4+r-1)}{1.2.3.\&c.\,r}x^r;$$

$\therefore$ the required coefficient

$$=\frac{4\cdot5\cdot6.\&c.\,(r+3)}{1.2.3.\&c.\,r}=\frac{4\cdot5\cdot6.\&c.\,r(r+1)(r+2)(r+3)}{1.2.3\cdot4\cdot5\cdot6.\&c.\,r}$$

$$=\frac{1}{6}(r+1)(r+2)(r+3).$$

23. Here $n=10$, $p=0$, $q=2$, $r=3$, $s=0$, $t=4$, $u=1$;

$\therefore$ the required coefficient

$$=\frac{\lfloor 10}{\lfloor 2\cdot\lfloor 3\cdot\lfloor 4}\cdot 2^2\cdot 3^3\cdot 5^4\cdot 6^1_{\bullet}=5103000000.$$

28. The middle term will obviously involve x^{12}, and to find the coefficient of x^{12} we have

$$\left.\begin{aligned}p+q+r&=12\\ q+2r&=12\end{aligned}\right\}\text{ for the equations of condition;}$$

$\therefore$ the coefficient of the middle term is

p	q	r
0	12	0
1	10	1
2	8	2
3	6	3
4	4	4
5	2	5
6	0	6

$$\lfloor 12\left\{\frac{1}{\lfloor 12}+\frac{1}{\lfloor 10}+\frac{1}{\lfloor 2.\lfloor 8.\lfloor 2}+\frac{1}{\lfloor 3.\lfloor 6.\lfloor 3}+\frac{1}{\lfloor 4.\lfloor 4.\lfloor 4}+\frac{1}{\lfloor 5.\lfloor 2.\lfloor 5}+\frac{1}{\lfloor 6.\lfloor 6}\right\}$$

$$=1+132+2970+18480+34650+16632+924$$
$$=73789;$$

$\therefore$ the middle term $=73789x^{12}$.

30. (1) $(a+b+c)^9 = a^9 + 9a^8(b+c) + 36a^7(b+c)^2 + \&c. + (b+c)^9.$

Whence we see that a^9 gives one term; $9a^8(b+c)$ gives two terms, $36a^7(b+c)^2$ gives *three* terms, and so on; so that the whole No. of terms will be given by the A.P. series

$$1+2+3+\&c.+10 = (1+10)\frac{10}{2} = 55.$$

(2) No. of terms in the expansion of any multinomial

$$(a_1+a_2+a_3+\ldots+a_r)^n \text{ is } \frac{r(r+1)(r+2)\ldots(r+n-1)}{1.2.3\ldots\ldots n}.$$

Here $n=4$, $r=4$;

$$\therefore \text{ the No. of terms} = \frac{4.5.6.7}{1.2.3.4} = 35.$$

31_1. The middle term being the $(n+1)^{\text{th}}$ term

$$= \frac{2n(2n-1)\ldots\ldots(n+1)}{1.2\ldots\ldots n} x^n . \frac{1}{x^n}$$

$$= \frac{1.3.5\ldots\ldots(2n-1)}{1.2.3\ldots\ldots n} . 2^n, \text{ (vide Ex. 53, } 28_1\text{)};$$

$$\therefore \left(x+\frac{1}{x}\right)^{2n} = x^{2n} + 2n.x^{2n-1}.\frac{1}{x} + 2n.\frac{2n-1}{2}.x^{2n-2}.\frac{1}{x^2} + \&c.$$

$$+\frac{1.3.5.\&c.(2n-1)}{1.2.3\ldots\ldots n}.2^n + \&c.$$

$$+2n\frac{(2n-1)}{2}x^2.\frac{1}{x^{2n-2}} + 2nx\frac{1}{x^{2n-1}} + \frac{1}{x^{2n}}$$

$$= \left(x^{2n}+\frac{1}{x^{2n}}\right) + 2n\left(x^{2n-2}+\frac{1}{x^{2n-2}}\right)$$

$$+\frac{2n(2n-1)}{1.2}\left(x^{2n-4}+\frac{1}{x^{2n-4}}\right) + \&c.$$

$$+\frac{1.3.5\ldots\ldots(2n-1)}{1.2.3.\&c.\ldots n}.2^n.$$

31_2. $$\left(\frac{1+2x}{1+x}\right)^n = \left(\frac{1+x}{1+2x}\right)^{-n} = \left(1-\frac{x}{1+2x}\right)^{-n}$$

$$= 1 + \frac{nx}{1+2x} + \frac{n(n+1)}{1.2}\left(\frac{x}{1+2x}\right)^2 + \&c.$$

$$+\frac{n(n+1)(n+2)\ldots(n+r-1)}{1.2.3\ldots\ldots r}\left(\frac{x}{1+2x}\right)^r + \&c.$$

32. The series

$$= 1 - \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{\left(\frac{1}{2}+1\right)}{2}\cdot\frac{1}{2^2} - \frac{1}{2}\cdot\frac{\left(\frac{1}{2}+1\right)}{2}\cdot\frac{\left(\frac{1}{2}+2\right)}{3}\cdot\frac{1}{2^3} + \&c.$$

$$= \left(1+\frac{1}{2}\right)^{-\frac{1}{2}} = \left(\frac{3}{2}\right)^{-\frac{1}{2}} = \left(\frac{2}{3}\right)^{\frac{1}{2}}.$$

35. The value

$$= a\left\{1 - n + \frac{n(n-1)}{2} - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} + \&c.\right\}$$

$$- b\left\{n - n(n-1) + \frac{n(n-1)(n-2)}{1\cdot 2} - \&c.\right\}$$

$$= a(1-1)^n - bn(1-1)^{n-1} = 0.$$

38. By question $\frac{n(n-1)}{1\cdot 2} = \frac{14}{9}$,

and $\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} = -\frac{7}{243}$;

$$\therefore \frac{(n-2)(n-3)}{3\cdot 4} = -\frac{7\times 9}{243\times 14} = -\frac{1}{54};$$

$$\therefore n^2 - 5n + 6 = -\frac{2}{9},$$

whence $n = \frac{7}{3}$.

39. $(1+x)^n = 1 + n_1 x + n_2 x^2 + n_3 x^3 + \&c. + n_{r-1}x^{r-1} + n_r x^r + \&c.$

and $(1+x)^p = 1 + p_1 x + p_2 x^2 + p_3 x^3 + \&c. + p_{r-1}x^{r-1} + p_r x^r + \&c.$

$\therefore$ $(1+x)^{n+p}$ = the product of these two series, also

$$(1+x)^{n+p} = 1 + (n+p)_1 x + (n+p)_2 x^2 + \&c. + (n+p)_r x^r + \&c.,$$

and these values of $(1+x)^{n+p}$ are identical; therefore the coefficients of the same powers of x are equal;

$$\therefore (n+p)_r = n_r + n_{r-1}\cdot p_1 + n_{r-2}\cdot p_2 + \&c. + n_1 p_{r-1} + p_r.$$

43. Let a, b, c, d be the coefficients of the $(r+1)^{\text{th}}$, $(r+2)^{\text{th}}$, $(r+3)^{\text{th}}$, $(r+4)^{\text{th}}$ terms respectively of an expanded binomial, whose index is n.

Then $a=\dfrac{n(n-1)\ldots(n-r+1)}{1.2\ldots\ldots r}$, $\quad b=\dfrac{n(n-1)\ldots(n-r)}{1.2\ldots\ldots(r+1)}$,

$$c=\frac{n(n-1)\ldots(n-r-1)}{1.2\ldots\ldots(r+2)}, \quad d=\frac{n(n-1)\ldots(n-r-2)}{1.2\ldots\ldots(r+3)}.$$

$$\text{Hence } \frac{b}{a}=\frac{n-r}{r+1}; \quad \therefore \frac{b}{a}+1=\frac{n+1}{r+1}.$$

$$\text{Similarly, } \frac{c}{b}+1=\frac{n+1}{r+2}, \text{ and } \frac{d}{c}+1=\frac{n+1}{r+3};$$

$$\therefore (r+1)\left(\frac{b}{a}+1\right)=n+1=(r+2)\left(\frac{c}{b}+1\right)=(r+3)\left(\frac{d}{c}+1\right);$$

$$\therefore r+1=(r+2)\frac{(b+c)a}{(a+b)b},$$

$$r+3=(r+2)\frac{(b+c)c}{(c+d)b}.$$

By addition, and dividing by $r+2$, we get

$$2=\frac{b+c}{b}\left\{\frac{a}{a+b}+\frac{c}{c+d}\right\};$$

$$\therefore 2b(ac+bc+ad+bd)=(b+c)(2ac+bc+ad);$$

$$\therefore (bc+ad)(b-c)=(b+c)(2ac)-2abc-2b^2d$$
$$=2(ac^2-b^2d).$$

45. Let $s=a^2+b^2$, $p=2ab$, and $P=(a+b)^p$, then

$$(s+p)=(a+b)^2, \text{ and } (s+p)^{\frac{p}{2}}=P;$$

$$\therefore P.P^{\frac{1}{2}}.P^{\frac{1}{4}}.\&c.=(s+p)^{\frac{p}{2}}.(s+p)^{\frac{p}{4}}.(s+p)^{\frac{p}{8}}.\&c.$$
$$=(s+p)^{\frac{p}{2}(1+\frac{1}{2}+\frac{1}{4}+\&c.)}$$
$$=(s+p)^p=s^p\left(1+\frac{p}{s}\right)^p;$$

$$\therefore P.P^{\frac{1}{2}}.P^{\frac{1}{4}}.P^{\frac{1}{8}}.\&c. \text{ in inf.}=s^p\left\{1+p\left(\frac{p}{s}\right)+\frac{p(p-1)}{1.2}.\left(\frac{p}{s}\right)^2 + \frac{p(p-1)(p-2)}{1.2.3}.\left(\frac{p}{s}\right)^3+\&c.\right\}.$$

INDETERMINATE COEFFICIENTS.

Ex. 54.

4. $x^4 - a^4 = (x^2 - a^2)(x^2 + a^2) = (x - a)(x + a)(x^2 + a^2).$

$$\text{Let} \quad \frac{1}{x^4 - a^4} = \frac{A}{x - a} + \frac{B}{x + a} + \frac{Cx + D}{x^2 + a^2},$$

then

$$1 = A(x + a)(x^2 + a^2) + B(x - a)(x^2 + a^2) + (Cx + D)(x^2 - a^2).$$

$$\text{Let } x = a, \quad \text{then } 1 = A(2a)(2a^2); \qquad \therefore A = \frac{1}{4a^3},$$

$$x = -a, \qquad 1 = B(-2a)(2a^2); \quad \therefore B = -\frac{1}{4a^3};$$

$$\therefore (Cx + D)(x^2 - a^2) = 1 - (x^2 + a^2)\left(\frac{x + a}{4a^3} - \frac{x - a}{4a^3}\right)$$

$$= 1 - \frac{x^2 + a^2}{2a^2} = -\frac{x^2 - a^2}{2a^2};$$

$$\therefore Cx + D = -\frac{1}{2a^2}; \text{ hence } C = 0, \ D = -\frac{1}{2a^2};$$

$$\therefore \frac{1}{x^4 - a^4} = \frac{1}{4a^3(x - a)} - \frac{1}{4a^3(x + a)} - \frac{1}{2a^2(x^2 + a^2)}.$$

8. $2x^3 - 14x^2 + 16x + 20 = 2x^2(x - 5) - 4(x^2 - 4x - 5)$

$$= 2(x - 5)(x^2 - 2x - 2).$$

$$\text{Let} \quad \frac{3x^2 - 8x + 16}{x^3 - 7x^2 + 8x + 10} = \frac{A}{x - 5} + \frac{Bx + C}{x^2 - 2x - 2}.$$

Then $3x^2 - 8x + 16 = A(x^2 - 2x - 2) + (Bx + C)(x - 5)$.

This equation is true for every value of x; let then $x = 5$,

$$5(3 \times 5 - 8) + 16 = A(25 - 10 - 2);$$

$$\therefore A = \frac{51}{13};$$

$$\therefore (Bx + C)(x - 5) = 3x^2 - 8x + 16 - \frac{51}{13}(x^2 - 2x - 2)$$

$$= -\frac{12x^2 + 2x - 310}{13}$$

$$= -\frac{(12x + 62)(x - 5)}{13};$$

$$\therefore Bx + C = -\frac{12x + 62}{13};$$

whence $B = -\dfrac{12}{13}, \quad C = -\dfrac{62}{13};$

$$\therefore \frac{3x^2 - 8x + 16}{2x^3 - 14x^2 + 16x + 20} = \frac{51}{26(x - 5)} - \frac{6x + 31}{13(x^2 - 2x - 2)}.$$

12. Let $\dfrac{3x - 2}{(x - 1)(x - 2)(x - 3)} = \dfrac{A}{x - 1} + \dfrac{B}{x - 2} + \dfrac{C}{x - 3}.$

Then $3x - 2 = A(x - 2)(x - 3) + B(x - 1)(x - 3)$
$+ C(x - 1)(x - 2).$

Let $x = 1$, then $1 = A(-1)(-2)$; $\therefore A = \dfrac{1}{2}$,

$x = 2$, ... $4 = B(1)(-1)$; $\therefore B = -4$,

$x = 3$, ... $7 = C(2)(1)$; $\therefore C = \dfrac{7}{2}$.

Now $\dfrac{A}{x - 1} = -\dfrac{1}{2}\left(\dfrac{1}{1 - x}\right) = -\dfrac{1}{2}(1 + x + x^2 + x^3 + \ldots + x^n + \ldots)$,

$$\frac{B}{x - 2} = 2\left(\frac{1}{1 - \frac{x}{2}}\right) = +2\left(1 + \frac{x}{2} + \frac{x^2}{2^2} + \frac{x^3}{2^3} + \ldots + \frac{x^n}{2^n} + \ldots\right),$$

$$\frac{C}{x - 3} = -\frac{7}{6}\left(\frac{1}{1 - \frac{x}{3}}\right) = -\frac{7}{6}\left(1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \ldots + \frac{x^n}{3^n} + \ldots\right).$$

Hence $\dfrac{3x - 2}{(x - 1)(x - 2)(x - 3)} = \dfrac{1}{3} + \dfrac{1}{9}x - \dfrac{7}{54}x^2 - \dfrac{95}{324}x^3 - \ldots$

$$+ \left(\frac{1}{2^{n-1}} - \frac{1}{2} - \frac{7}{6} \cdot \frac{1}{3^n}\right)x^n + \ldots$$

Reversion of Series.

Ex. 55.

4. Let $y - a = z = bx + cx^2 + dx^3$.

Assume $x = Az + Bz^2 + Cz^3 + Dz^4 + \dots$.

$$\begin{aligned}
\text{Then } bx &= bAz + bBz^2 + bCz^3 + bDz^4 + \dots \\
cx^2 &= \quad + cA^2z^2 + 2cABz^3 + c(B^2 + AC)z^4 + \dots \\
dx^3 &= \quad\quad + dA^3z^3 + 3dA^2Bz^4 + \dots;
\end{aligned}$$

$$\therefore\ z = bAz + (bB + cA^2)z^2 + (bC + 2cAB + dA^3)z^3 + \dots.$$

Hence $bA = 1; \qquad \therefore\ A = \frac{1}{b}$,

$$bB = -cA^2; \quad \therefore\ B = -\frac{c}{b^3},$$

$$bC = -(2cAB + dA^3) = +2c.\frac{c}{b^4} - \frac{d}{b^3};$$

$$\therefore\ C = \frac{2c - bd}{b^5},$$

$$bD = -(cB^2 + cAC + 3dA^2B);$$

$$\therefore\ D = -\frac{5c^3 - 5bcd}{b^7},$$

&c. = &c.;

$$\therefore\ x = \frac{1}{b}(y-a) - \frac{c}{b^3}(y-a)^2 + \frac{2c^2 - bd}{b^5}(y-a)^3$$

$$-\frac{5c(c^2 - bd)}{b^7}(y-a)^4 + \text{\&c.}$$

11. $x^3 - 3x + y = 0$.

When $y = 0$, then $x(x^2 - 3) = 0$;

$\therefore\ x = 0$, and $x = \sqrt{3}$.

We shall have a series corresponding to each of these values of x.

(1) Let $x = Ay + By^2 + Cy^3 + Dy^4 + Ey^5 + \dots$

Then $y = \quad y$

$$-3x = -3Ay - 3By^2 - 3Cy^3 - 3Dy^4 - 3Ey^5 - \dots$$

$$x^3 = \qquad +A^3y^3 + 3A^2By^4 + (3A^2C + 3AB^2)y^5 + \dots;$$

$$\therefore 0 = (1-3A)y - 3By^2 + (A^3 - 3C)y^3 + 3(A^2B - D)y^4 + \dots;$$

$$\therefore A = \frac{1}{3},\quad B = 0,\quad C = \frac{A^3}{3} = \frac{1}{3^4},\quad D = 0,\quad E = A^2C = \frac{1}{3^6},\ \&c. \dots$$

$$\therefore x = \frac{y}{3} + \frac{y^3}{3^4} + \frac{y^5}{3^6} + \dots$$

(2) Let $x = a + by + cy^2 + dy^3 + ey^4 + \dots$

Then $x^3 = a^3 + 3a^2y(b + cy + dy^2 + ey^3 + \dots)$

$$+ 3ay^2\{b + y(c + dy + \dots)\}^2 + y^3\{b + y(c + \dots)\}^3$$

$$= a^3 + 3a^2by + 3a^2cy^2 + 3a^2dy^3 + 3a^2ey^4 + \dots$$

$$+ 3ab^2y^2 + 6abcy^3 + (6abd + 3ac^2)y^4 + \dots$$

$$+ b^3y^3 + 3b^2cy^4 + \dots$$

$$-3x = -3a - 3by - 3cy^2 - 3dy^3 - 3ey^4 - \dots$$

$$+y = \quad +y;$$

$$\therefore 0 = (a^3 - 3a) + (3a^2b - 3b + 1)y + (3a^2c + 3ab^2 - 3c)y^2 + \dots;$$

$$\therefore a = \sqrt{3},\quad b = -\frac{1}{6},\quad c = -\frac{\sqrt{3}}{72},\quad d = -\frac{1}{162},\ \&c. = \&c.$$

$$\therefore x = \sqrt{3} - \frac{1}{6}y - \frac{\sqrt{3}}{72}y^2 - \frac{1}{162}y^3 - \frac{35\sqrt{3}}{5184}y^4 - \dots$$

SUMMATION OF SERIES.

Ex. 56.

2. Assume

$$1^2+4^2+7^2+\dots+(3n-2)^2=A+Bn+Cn^2+Dn^3+En^4+\dots$$

Change n into $n+1$;

$$\therefore\ 1^2+4^2+7^2+\dots+(3n-2)^2+(3n+1)^2=A+B(n+1)$$
$$+C(n+1)^2+D(n+1)^3+E(n+1)^4+\dots$$

By subtraction

$$9n^2+6n+1=B+C(2n+1)+D(3n^2+3n+1)$$
$$+E(4n^3+6n^2+4n+1)+\dots$$

Equate the coefficients of the same powers of n, then $E=0$, and the coefficient of every term after E is $=0$; then

$$3D=9,\quad 3D+2C=6,\quad D+C+B=1;$$

$$\therefore\ D=3,\quad C=-\frac{3}{2},\quad B=-\frac{1}{2}.$$

Hence $1^2+4^2+7^2+\dots+(3n-2)^2=A-\dfrac{n}{2}-\dfrac{3n^2}{2}+3n^3.$

To find A; the last equation is true for every integral value of n;

let then $n=1$; $\therefore\ A=0$;

$$\therefore\ 1^2+4^2+7^2+\dots+(3n-2)^2=\frac{n}{2}(6n^2-3n-1).$$

5. Assume

$$1\cdot 2^2+2\cdot 3^2+3\cdot 4^2+\dots+n(n+1)^2$$
$$=A+Bn+Cn^2+Dn^3+En^4+Fn^5+\dots$$

Then $1\cdot 2^2+2\cdot 3^2+3\cdot 4^2+\dots+n(n+1)^2+(n+1)(n+2)^2$

$$=A+B(n+1)+C(n+1)^2$$
$$+D(n+1)^3+E(n+1)^4$$
$$+F(n+1)^5+\dots$$

$$\therefore\ n^3+5n^2+8n+4=B+C(2n+1)+D(3n^2+3n+1)$$
$$+E(4n^3+6n^2+4n+1)$$
$$+F(5n^4+10n^3+\dots)+\dots$$

Hence $F=0$, and the coefficient of every higher power of $n=0$,

$$4E=1,\ 6E+3D=5,\ 4E+3D+2C=8,\ E+D+C+B=4;$$

$$\therefore\ E=\frac{1}{4},\quad D=\frac{7}{6},\quad C=\frac{7}{4},\quad B=\frac{5}{6};$$

$$\therefore\ 1.2^2+2.3^2+3\cdot 4^2+\dots+n(n+1)^2$$
$$=A+\frac{5n}{6}+\frac{7n^2}{4}+\frac{7n^3}{6}+\frac{n^4}{4};\text{ when } n=1,\ A=0$$
$$=\frac{n}{12}(n+1)(n+2)(3n+5).$$

Ex. 57.

1. Let $S=1+\frac{1}{2}+\frac{1}{3}+\&c.+\frac{1}{n}+\frac{1}{n+1};$

$$\therefore\ S-1=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\&c.+\frac{1}{n+1};$$

$$\therefore\ 1=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3\cdot 4}+\&c.+\frac{1}{n(n+1)}+\frac{1}{n+1};$$

therefore the required sum

$$=1-\frac{1}{n+1}=\frac{n}{n+1},$$

and if $n=\infty$, the sum $=1$.

4. Assume $\frac{1}{3}+\frac{1}{8}+\frac{1}{13}+\&c.+\frac{1}{5n-2}+\frac{1}{5n+3}=S;$

$$\therefore\ \frac{1}{8}+\frac{1}{13}+\frac{1}{18}+\&c.+\frac{1}{5n+3}=S-\frac{1}{3};$$

$$\therefore\ 5\left\{\frac{1}{3.8}+\frac{1}{8.13}+\&c.+\frac{1}{(5n-2)(5n+3)}\right\}+\frac{1}{5n+3}=\frac{1}{3};$$

therefore the required sum

$$=\frac{1}{15}-\frac{1}{5(5n+3)}=\frac{n}{3(5n+3)},$$

and if $n=\infty$, the sum $=\dfrac{1}{15}$.

8. The n^{th} term $=\dfrac{n+1}{(2n-1)(2n+1)(2n+3)(2n+5)}$

$$=\frac{1}{6}\left\{\frac{n+1}{(2n-1)(2n+1)(2n+3)}-\frac{n+1}{(2n+1)(2n+3)(2n+5)}\right\};$$

$$\therefore S_n=\frac{1}{6}\left\{\frac{2}{1.3.5}+\frac{3}{3.5.7}+\frac{4}{5.7.9}+\ldots+\frac{n+1}{(2n-1)(2n+1)(2n+3)}\right.$$
$$\left.-\frac{2}{3.5.7}-\frac{3}{5.7.9}-\ldots-\frac{n}{(2n-1)(2n+1)(2n+3)}-\frac{n+1}{(2n+1)(2n+3)(2n+5)}\right\}$$

$$=\frac{1}{90}+\frac{1}{6}\left\{\frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+\ldots+\frac{1}{(2n-1)(2n+1)(2n+3)}\right\}-\frac{n+1}{6(2n+1)(2n+3)(2n+5)}$$

$$=\frac{1}{90}+\frac{1}{6}\left\{\frac{1}{12}+\frac{1}{4(2n+1)(2n+3)}\right\}-\frac{n+1}{6(2n+1)(2n+3)(2n+5)}$$

$$=\frac{1}{90}+\frac{1}{72}-\frac{1}{24(2n+1)(2n+3)}\left\{\frac{4n+4}{2n+5}+\frac{2n+5}{2n+5}\right\}$$

$$=\frac{1}{40}-\frac{1}{8(2n+1)(2n+5)}=\frac{n(n+3)}{10(2n+1)(2n+5)}.$$

Hence $S_\infty=\dfrac{1}{40}$; putting $n=\infty$.

Ex. 58.

1. Assume $1+\dfrac{x}{2}+\dfrac{x^2}{3}+\ldots+\dfrac{x^{n-1}}{n}+\dfrac{x^n}{n+1}+\dfrac{x^{n+1}}{n+2}=S.$

Multiply both sides by $x^2 - 1$; then

$$\left.\begin{array}{l} \dfrac{x^2}{1}+\dfrac{x^3}{3}+\dfrac{x^4}{4}+\ldots+\dfrac{x^{n+1}}{n}+\dfrac{x^{n+2}}{n+1}+\dfrac{x^{n+3}}{n+2} \\ -1-\dfrac{x}{2}-\dfrac{x^2}{3}-\dfrac{x^3}{4}-\dfrac{x^4}{5}-\ldots-\dfrac{x^{n+1}}{n+2} \end{array}\right\} = (x^2-1)\,S;$$

$$\therefore\ -1-\frac{x}{2}+\frac{2x^2}{1\cdot 3}+\frac{2x^3}{2\cdot 4}+\frac{2x^4}{3\cdot 5}+\ldots+\frac{2x^{n+1}}{n(n+2)}$$

$$= (x^2-1)\,S-\frac{x^{n+2}}{n+1}-\frac{x^{n+3}}{n+2}.$$

Let $x=1$; then we get

$$\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{n(n+2)}$$

$$=\frac{1}{2}\left\{\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right\}=\frac{n(3n+5)}{4(n+1)(n+2)}.$$

When n becomes infinite, the sum

$$=\frac{3+\dfrac{5}{n}}{4\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{2}{n}\right)}=\frac{3}{4}.$$

4. Assume $1+\dfrac{x}{2}+\dfrac{x^2}{3}+\dfrac{x^3}{4}+\ldots+\dfrac{x^{n-1}}{n}+\dfrac{x^n}{n+1}=S.$

Multiply both sides by $p+qx$; then

$$\left.\begin{array}{l} p+\dfrac{px}{2}+\dfrac{px^2}{3}+\dfrac{px^3}{4}+\ldots+\dfrac{px^{n-1}}{n}+\dfrac{px^n}{n+1} \\ +qx+\dfrac{qx^2}{2}+\dfrac{qx^3}{3}+\ldots+\dfrac{qx^{n-1}}{n-1}+\dfrac{qx^n}{n}+\dfrac{qx^{n+1}}{n+1} \end{array}\right\} = (p+qx)\,S;$$

$$\therefore\ p+\frac{p+2q}{1\cdot 2}\,x+\frac{2p+3q}{2\cdot 3}\cdot x^2+\ldots+\frac{np+(n+1)q}{n(n+1)}$$

$$=(p+qx)\,S-\frac{qx^{n+1}}{n+1}.$$

If $p+qx=0$, $x=\frac{1}{2}$; we may have $p=-1$, $q=2$; then

$$\frac{3}{1.2}\cdot\frac{1}{2}+\frac{4}{2.3}\cdot\frac{1}{2^2}+\frac{5}{3.4}\cdot\frac{1}{2^3}+\dots+\frac{n+2}{n(n+1)}\cdot\frac{1}{2^n}$$
$$=1-\frac{1}{n+1}\cdot\frac{1}{2^n}.$$

When $n=\infty$, the sum $=1$.

6. Assume $1+\frac{x}{2}+\frac{x^2}{3}+\dots+\frac{x^{n-1}}{n}+\frac{x^n}{n+1}+\frac{x^{n+1}}{n+2}=S.$

Multiply both sides by $p+qx+rx^2$; then

$$\left.\begin{aligned}&p+\frac{px}{2}+\frac{px^2}{3}+\frac{px^3}{4}+\dots+\frac{px^{n+1}}{n+2}\\&\quad+qx+\frac{qx^2}{2}+\frac{qx^3}{3}+\dots+\frac{qx^{n+1}}{n+1}+\frac{qx^{n+2}}{n+2}\\&\qquad+rx^2+\frac{rx^3}{2}+\dots+\frac{rx^{n+1}}{n}+\frac{rx^{n+2}}{n+1}+\frac{rx^{n+3}}{n+2}\end{aligned}\right\}$$
$$=(p+qx+rx^2)\,S.$$

Let $p+qx+rx^2=0$, and $x=\frac{1}{2}$; then

$$\left.\begin{aligned}4p+2q+r&=0.\\ \text{Also }2p+3q+6r&=19,\\ \text{and }6p+8q+12r&=28;\end{aligned}\right\}$$

$$\therefore\ p=2,\quad q=-7,\quad r=6.$$

Whence $2+(-6)\frac{1}{2}+\frac{19}{1.2.3}\cdot\frac{1}{4}+\frac{28}{2.3.4}\cdot\frac{1}{8}+\dots$

$$+\frac{2n(n+1)-7n(n+2)+6(n+1)(n+2)}{n(n+1)(n+2)}\cdot\frac{1}{2^{n+1}}$$
$$=\frac{7}{n+2}\cdot\frac{1}{2^{n+2}}-6\left\{\frac{1}{n+1}+\frac{1}{2(n+2)}\right\}\cdot\frac{1}{2^{n+2}};$$

$$\therefore\ \frac{19}{1.2.3}\cdot\frac{1}{4}+\frac{28}{2.3.4}\cdot\frac{1}{8}+\frac{39}{3.4.5}\cdot\frac{1}{16}+\dots\text{ to }n\text{ terms}$$
$$=1-\frac{n+4}{(n+1)(n+2)}\cdot\frac{1}{2^{n+1}}.$$

When $n=\infty$, the sum $=1$.

Ex. 59.

1. Let $1-px-qx^2$ be the *scale of relation.*

Assume $S=1+2x+3x^2+4x^3+\dots+nx^{n-1}$,

then $-pxS=-px-2px^2-3px^3-\dots-(n-1)px^{n-1}-npx^n$,

and $-qx^2S=-qx^2-2qx^3-\dots-(n-2)qx^{n-1}-(n-1)qx^n-nqx^{n+1}$;

$$\therefore (1-px-qx^2)S=1+(2-p)x+(3-2p-q)x^2+(4-3p-2q)x^3+\dots+\{n-(n-1)p-(n-2)q\}x^{n-1}-\{np+(n-1)q\}x^n-nqx^{n+1}.$$

Hence $\left.\begin{array}{l}2p+q=3\\ 3p+2q=4\end{array}\right\}$; $\therefore p=2,\quad q=-1$;

$$\therefore (1-2x+x^2)S=1-(n+1)x^n+nx^{n+1};$$

$$\therefore S=\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}.$$

5. Let $\frac{x}{3}=-z$; and $1-pz-qz^2$ be the *scale of relation.*

Assume $S=1+7z+17z^2+55z^3+\dots\dots$ to infinity,

then $-pzS=-pz-7pz^2-17pz^3-\dots\dots$

$-qz^2S=-qz^2-7qz^3-\dots\dots$;

$$\therefore (1-pz-qz^2)S=1+(7-p)z+(17-7p-q)z^2+(55-17p-7q)z^3+\dots\dots$$

Hence $\left.\begin{array}{l}7p+q=17\\ 17p+7q=55\end{array}\right\}$; $\therefore p=2,\ q=3$;

$$\therefore S=\frac{1+5z}{1-2z-3z^2}=\frac{1+5z}{(1-3z)(1+z)}=\frac{2}{1-3z}-\frac{1}{1+z}=\frac{2}{1+x}-\frac{1}{1-\frac{x}{3}}$$

$$=2(1-x+x^2-x^3+\dots)-\left(1+\frac{x}{3}+\frac{x^2}{9}+\frac{x^3}{27}+\dots\right).$$

Ex. 60.

1. The orders of differences are

$$\begin{array}{r} 4,\ 10,\ 20,\ 35,\ 56,\ \&c.\ \text{or}\ d_1=4, \\ 6,\ 10,\ 15,\ 21,\ \&c.\ \text{or}\ d_2=6, \\ 4,\ 5,\ 6,\ \&c.\ \text{or}\ d_3=4, \\ 1,\ 1,\ \&c.\ \text{or}\ d_4=1, \\ 0,\ \&c.\ \text{or}\ d_5=0; \end{array}$$

$$\therefore \text{ the 12th term} = 1+11.4+\frac{11.10}{1.2}.6+\frac{11.10.9}{1.2.3}.4+\frac{11.10.9.8}{1.2.3.4}$$
$$= 1+44+330+660+330=1365.$$

12. By giving to n the values 1, 2, 3, &c. successively, we find the series to be $1^2.1+2^2.4+3^2.7+4^2.10+5^2.13+\&c.$, whence $d_1=15$, $d_2=32$, $d_3=18$, $d_4=0$;

$$\therefore \text{ the sum} = 25+\frac{25.24}{1.2}.15+\frac{25.24.23}{1.2.3}.32+\frac{25.24.23.22}{1.2.3.4}.18$$
$$=25+4500+78600+227700$$
$$=305825.$$

Interpolation of Series.

Ex. 61.

1. If a, b, c, &c. represent the 1st, 2nd, 3rd, &c. terms respectively of the series, we have

$$a-6b+15c-20d+15e-6f+g=0,$$

whence e, the 5^{th} term $=\dfrac{20d+6(b+f)-15c-(a+g)}{15}$

$$=\frac{1000+900-450-220}{15}$$
$$=\frac{1230}{15}=82.$$

8. If we suppose the 4^{th} order of differences to vanish, then

$$a-4b+6c-4d+e=0;$$

$$\therefore\ d=\frac{a+e+6c}{4}-b=\frac{\sqrt{19}+\sqrt{23}+6\sqrt{21}}{4}-\sqrt{20}=4{\cdot}6904101.$$

5. Here $d_1 = 3$, $d_2 = 3$, $d_3 = 1$, $d_4 = 0$.

Then $y = a + (x-1)d_1 + \frac{(x-1)(x-2)}{1.2}d_2$

$$+ \frac{(x-1)(x-2)(x-3)}{1.2.3}d_3.$$

Let $x = 1\frac{1}{4}$, then $y = 1 + \frac{3}{4} - \frac{9}{32} + \frac{7}{128} = \frac{195}{128}$,

$x = 1\frac{1}{2}$, ... $y = 1 + \frac{3}{2} - \frac{3}{8} + \frac{1}{16} = \frac{280}{128}$,

$x = 1\frac{3}{4}$, ... $y = 1 + \frac{9}{4} - \frac{9}{32} + \frac{5}{128} = \frac{385}{128}$,

$x = 2\frac{1}{4}$, ... $y = 1 + \frac{15}{4} + \frac{15}{32} - \frac{5}{128} = \frac{663}{128}$;

$\therefore$ the terms are,

$$1,\ \frac{195}{128},\ \frac{280}{128},\ \frac{385}{128},\ 4,\ \frac{663}{128},\ \&c.$$

CHANCES OR PROBABILITIES.

Ex. 62.

1. The chance of throwing an ace the first time $= \frac{1}{6}$; and the chance of *not* throwing an ace the second time $= \frac{5}{6}$; therefore the chance of throwing an ace in the first throw only is

$$\frac{1}{6} \times \frac{5}{6} = \frac{5}{36}.$$

4. For the balls to form pairs, one of the first 7 letters must be drawn from the first bag; the chance of this is $\frac{7}{12}$. Then the chance of drawing the same letter from the second bag $= \frac{1}{7}$;

$$\therefore \text{ the required probability} = \frac{7}{12} \times \frac{1}{7} = \frac{1}{12}.$$

5. The chance of drawing a white ball each time $=\frac{1}{2}$; therefore the chance of doing so 11 times successively

$$=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\text{\&c. to 11 factors}=\frac{1}{2^{11}}=\frac{1}{2048}.$$

9. The combinations of 10 things, 3 at a time

$$=\frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3}=120.$$

The chance of drawing

2 gs. and 1 sov.	$=\frac{3}{120}$;	its value	$=\frac{1}{120}\times 186$ shs.
2 gs. and 1 sh.	$=\frac{5}{120}$;	....	$=\frac{1}{120}\times 215$ ...
1 g. 1 sov. and 1 sh.	$=\frac{2\times 3\times 5}{120}$;	....	$=\frac{1}{120}\times 1260$ ...
2 sov. and 1 g.	$=\frac{6}{120}$;	....	$=\frac{1}{120}\times 366$ ...
2 sov. and 1 sh.	$=\frac{15}{120}$;	....	$=\frac{1}{120}\times 615$ ...
2 sh. and 1 g.	$=\frac{20}{120}$;	....	$=\frac{1}{120}\times 460$ ...
2 sh. and 1 sov.	$=\frac{30}{120}$;	....	$=\frac{1}{120}\times 660$ ...
3 sh.	$=\frac{10}{120}$;	....	$=\frac{1}{120}\times 30$ ...
3 sov.	$=\frac{1}{120}$;	....	$=\frac{1}{120}\times 60$ ...

Hence, value of expectation $=\frac{1}{120}\times 3852s.=32\cdot 1=£1.12s.1\cdot 2d.$

11. The chance of third ball being white $=\frac{1}{2}$.

Combinations of 3 things, 2 at a time $=\frac{3\cdot 2}{1\cdot 2}=3$.

Of these, the chance that a particular couple is drawn $=\frac{1}{3}$;

$\therefore$ the required chance = the compound chance

$$=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}.$$

13. The sum of the favorable and unfavorable events equal the No. of combinations of 25 things, 10 at a time

$$=\frac{25 \cdot 24 \ldots 16}{1 \cdot 2 \ldots 10}.$$

The combinations of 5 black balls, 3 at a time $=\frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}$,

.... 20 white balls, 7 .... $=\frac{20 \cdot 19 \ldots 14}{1 \cdot 2 \ldots 7}$;

$\therefore$ the sum of the favorable events $=\frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} \times \frac{20 \cdot 19 \ldots 14}{1 \cdot 2 \ldots 7}$;

$\therefore$ the probability required

$$=\frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} \times \frac{20 \cdot 19 \ldots 14}{1 \cdot 2 \ldots 7} \div \frac{25 \cdot 24 \ldots 16}{1 \cdot 2 \ldots 10}=\frac{60}{253}.$$

14. The No. of combinations of 52 things, 13 at a time

$$=\frac{52 \cdot 51 \cdot 50 \ldots 40}{1 \cdot 2 \cdot 3 \ldots 13}.$$

The No. of combinations of 48 things, 12 at a time

$$=\frac{48 \cdot 47 \cdot 46 \ldots 37}{1 \cdot 2 \cdot 3 \ldots 12}.$$

With each of the latter set, only one of the 4 aces may be combined; $\therefore$ chance first required

$$=\frac{48 \cdot 47 \ldots 37}{1 \cdot 2 \ldots 12} \times 4 \div \frac{52 \cdot 51 \ldots 40}{1 \cdot 2 \ldots 13}$$

$$=\frac{39 \cdot 38 \cdot 37 \cdot (13 \times 4)}{52 \cdot 51 \cdot 50 \cdot 49}=\frac{9139}{20825}=\frac{703}{1602}=\frac{7}{16} \text{ nearly.}$$

Also, chance of dealing one ace to each person

$$=\frac{7}{16} \times \frac{1}{4}=\frac{7}{64} \text{ nearly.}$$

16. Chance of dipping into the 1st or 2nd urn $=\frac{1}{2}$;

$\therefore$ the chance of drawing white from 1st urn $=\frac{1}{2} \times \frac{5}{9}$;

and the chance of drawing white from 2nd urn $=\frac{1}{2} \times \frac{5}{3}$;

$$\therefore \text{ the required chance }=\frac{1}{2}\left(\frac{5}{9}+\frac{3}{5}\right)=\frac{26}{45}.$$

18. The even chance of throwing an ace in x times

$$=\frac{1}{6}+\frac{1}{6}\left(\frac{5}{6}\right)+\frac{1}{6}\left(\frac{5}{6}\right)^2+\ldots.+\frac{1}{6}\left(\frac{5}{6}\right)^{x-1};$$

$$\therefore\ \frac{1}{2}=1-\left(\frac{5}{6}\right)^x;$$

$$\therefore\ (1\cdot 2)^x=2,\quad x=\frac{\log 2}{\log 1\cdot 2}=3\cdot 8018.$$

19. Let a, b be the numbers of white and black balls respectively; n the number of bags.

The probability of drawing a white ball from each bag $=\dfrac{a}{a+b}$.

.... black .... $=\dfrac{b}{a+b}$.

The prob^y. that the n balls drawn will all be white $=\left(\dfrac{a}{a+b}\right)^n$.

The probability that the n balls drawn will be

$$(n-1)\text{ white, and } 1 \text{ black}=n\frac{a^{n-1}b}{(a+b)^n},$$

$$(n-2)\ \ldots\ldots,\text{ and } 2\ \ldots\ldots=\frac{n(n-1)}{1\,.\,2}\cdot\frac{a^{n-2}b^2}{(a+b)^n},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots$$

$$(n-r)\ \ldots\ldots,\text{ and } r\ \ldots\ldots=\frac{n(n-1)\ldots(n-r+1)}{1\,.\,2\ldots r}\cdot\frac{a^{n-r}b^r}{(a+b)^n}.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The most probable number corresponds with that expression in the above series which is greatest;

if now $\dfrac{n(n-1)\ldots(n-r+1)}{1\,.\,2\ldots r}\cdot\dfrac{a^{n-r}b^r}{(a+b)^n}$ be the greatest,

then $\dfrac{n(n-1)\ldots(n-r+1)}{1\,.\,2\ldots r}a^{n-r}b^r>\dfrac{n(n-1)\ldots(n-r)}{1\,.\,2\ldots(r+1)}\,.\,a^{n-r-1}.b^{r+1};$

$$\therefore\ \frac{a}{b}>\frac{n-r}{r+1};\quad \therefore\ r>\frac{nb-a}{a+b}.$$

Also $\dfrac{n(n-1)\dots(n-a+1)}{1\,.\,2\dots r}a^{n-r}b^r > \dfrac{n(n-1)\dots(n-r+2)}{1\,.\,2\dots(r-1)}a^{n-r+1}b^{r-1}$;

$$\therefore \frac{n-r+1}{r}\cdot\frac{b}{a} > 1; \quad \therefore r < \frac{nb+b}{a+b}.$$

Here $n = 12$, $a = 7$, $b = 2$; $\therefore r > \dfrac{17}{9}$, and $< \dfrac{26}{9}$.

Since r is an integer, it must $= 2$;

$\therefore$ the most probable number of white balls drawn $= n - r = 10$.

20. The number of ways in which this can be done is expressed by the coefficient of x^{16} in the expansion of

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^4,$$

which coefficient, found by the Multinomial Theorem, is

$$6 + 24 + 12 + 12 + 12 + 1 + 12 + 24 + 4 + 6 + 12 = 125;$$

but there are $6^4 = 1296$ *possible* throws, hence the *required* chance $= \dfrac{125}{1296}$.

Scales of Notation.

Ex. 63.

1.

$$\begin{array}{r|l}
3 & 5381 \\
3 & 1793 \dots\dots 2 \\
3 & 597 \dots\dots 2 \\
3 & 199 \dots\dots 0 \\
3 & 66 \dots\dots 1 \\
3 & 22 \dots\dots 0 \\
3 & 7 \dots\dots 1 \\
 & 2 \dots\dots 1
\end{array}
\qquad\qquad
\begin{array}{r|l}
9 & 5381 \\
9 & 597 \dots\dots 8 \\
9 & 66 \dots\dots 3 \\
 & 7 \dots\dots 3
\end{array}$$

$\therefore$ the No. is 21101022. $\qquad$ $\therefore$ the No. is 7338.

3.

$$\begin{array}{r l}
34402 & \\
\underline{\quad 5} & \\
103_4, & \text{since } 19_{10} = 103_4 \\
\underline{\quad 5} & \\
1203 & \\
\underline{\quad 5} & \\
13233 & \\
\underline{\quad 5} & \\
212231 & \qquad \text{Hence } (34402)_5 = (212231)_4.
\end{array}$$

9.

$$\begin{array}{r}
tt4\,)\,95088918\,(\,t4tee \\
\underline{9074} \\
4548 \\
\underline{3754} \\
9e49 \\
\underline{9074} \\
t951 \\
\underline{9e58} \\
9e58 \\
\underline{9e58}
\end{array}$$

11.

$$\begin{array}{r r}
 & 2\dot{5}4\dot{0}0\dot{5}4\dot{4}\,(\,4112 \\
 & \underline{24} \\
121\,) & 140 \\
 & \underline{121} \\
1221\,) & 1505 \\
 & \underline{1221} \\
12222\,) & 24444 \\
 & \underline{24444}
\end{array}
\qquad
\begin{array}{r r}
 & 3\dot{2}e\dot{7}5\dot{7}2\dot{1}\,(\,62te \\
 & \underline{30} \\
102\,) & 2e7 \\
 & \underline{204} \\
104t\,) & e357 \\
 & \underline{t404} \\
1058e\,) & e5321 \\
 & \underline{e5321}
\end{array}$$

$\therefore$ the sq. root $= 4112$. $\qquad$ $\therefore$ the sq. root $= 62te$.

14. Let r be the radix of the scale,

then $2r^4 + 3r^2 + 5 = 4954$,

$$\text{or } r^4 + \frac{3}{2}r^2 = \frac{4949}{2},$$

whence $r = 7$.

16. Here we have to express 1719 by means of the terms of the series 1, 2, 2^2, 2^3, &c.; and to transform 1719 into the binary scale, we have

$$\begin{array}{r} 2\,\underline{)1719} \\ 2\,\underline{)859}\ldots\ldots1 \\ 2\,\underline{)429}\ldots\ldots1 \\ 2\,\underline{)214}\ldots\ldots1 \\ 2\,\underline{)107}\ldots\ldots0 \\ 2\,\underline{)53}\ldots\ldots1 \\ 2\,\underline{)26}\ldots\ldots1 \\ 2\,\underline{)13}\ldots\ldots0 \\ 2\,\underline{)6}\ldots\ldots1 \\ 2\,\underline{)3}\ldots\ldots0 \\ 1\ldots\ldots1 \end{array}$$

whence the number 1719 is equivalent to 11010110111 in the binary scale, which is expressed by $1+2+2^2+2^4+2^5+2^7+2^9+2^{10}$; and therefore it will be necessary to select the weights 1 lb., 2 lbs., 2^2 lbs., 2^4 lbs., 2^5 lbs., 2^7 lbs., 2^9 lbs., and 2^{10} lbs.

18. Let a_0, a_1, a_2, a_3, &c. be the digits, so that

$$N = a_0 + 10a_1 + 10^2a_2 + 10^3a_3 + \&c.$$

of which the general term is $10^m a_m$; then with the same digits differently arranged to make N', the general term containing a_m will be $10^n a_m$; and therefore the general term of $N \sim N'$ will be

$$(10^m - 10^n)\,a_m = 10^n\,(10^{m-n} - 1)\,a_m \text{ if } m > n,$$

$$\text{or } = 10^m\,(10^{n-m} - 1)\,a_m \text{ if } n > m;$$

since this term is divisible by $10 - 1$;

$\therefore$ every term in $N \sim N'$ is divisible by 9.

LOGARITHMS.

Ex. 70.

1. $\log 5{\cdot}4 = 3\log 3 + \log 2 - \log 10 = 1{\cdot}431363 + {\cdot}301030 - 1 = {\cdot}732939.$

$$\log 17{\cdot}5 = \log 7 + 2\log 15 - 2\log 3 - \log 10$$
$$= {\cdot}845098 + 2{\cdot}352182 - {\cdot}954242 - 1$$
$$= 1{\cdot}243038.$$

3. $$256 = 2^8 = \left(2^{\frac{3}{2}}\right)^{\frac{16}{3}};$$

$$\therefore \text{ the required } \log 256 = \frac{16}{3} = 5\cdot\dot{3}.$$

Exponential Equations.

Ex. 71.

1. $$x \log 20 = \log 100;$$

$$\therefore\ x = \frac{\log 100}{\log 20} = \frac{2}{1\cdot3010300} = 1\cdot537.$$

7. $$(3\cdot5)\,x = \frac{\log 1000}{\log 4\cdot5} = \frac{3}{\cdot6532125};$$

$$\therefore\ x = \frac{6}{4\cdot57224875} = 1\cdot312.$$

13. $$\left(\frac{17}{14}\right)^x = \frac{29}{21};\quad \therefore\ x = \frac{\log 29 - \log 21}{\log 17 - \log 14} = 1\cdot6624;$$

$$\therefore\ \log y = 1\cdot6624 \times \log 14 - \log 63;$$

$$\therefore\ y = 1\cdot2764.$$

Interest and Annuities.

Ex. 72.

1. $$\text{Amount } M = P\left(1 + \frac{r}{q}\right)^{nq},$$

$$\text{and } P = 660,\quad r = \cdot05,\quad n = 6,\quad q = 4;$$

$$\therefore M = 660\,(1 + \cdot0125)^{24};$$

$$\therefore \log M = \log 660 + 24 \log 1\cdot0125$$

$$= 2\cdot9490239 = \log 889\cdot25;$$

$$\therefore \text{ the amount} = 889\cdot25 = £889.\ 5s.$$

8. Here $3P = P\left(1 + \frac{r}{q}\right)^{nq}$, and $q = 2$, $r = \cdot 03$;

$$\therefore (1\cdot015)^{2n} = 3, \text{ or } n \times 2 \log 1\cdot015 = \log 3;$$

$$\therefore n = \frac{\log 3}{2 \log 1\cdot015} = 36\cdot894.$$

12. Let x, y, z be the present values of the sums which A, B, C respectively have to receive, then

$$x + y + z = 3500 \dots\dots\dots\dots (1),$$

$$\text{and } x \times (1\cdot04)^7 = y \times (1\cdot04)^9 = z \times (1\cdot04)^{12};$$

$$\therefore y = \frac{x}{(1\cdot04)^2}, \quad z = \frac{x}{(1\cdot04)^5};$$

$$\therefore \text{from } (1),\ x\left\{1 + \frac{1}{(1\cdot04)^2} + \frac{1}{(1\cdot04)^5}\right\} = 3500;$$

$$\therefore x = \frac{3500 \times (1\cdot04)^5}{1 + (1\cdot04)^3 + (1\cdot04)^5} = \frac{4258\cdot282}{3\cdot3415159},$$

whence $x = £1274.\ 7s.\ 1\frac{1}{2}d.$ nearly;

$$\therefore y = \frac{x}{(1\cdot04)^2} = \frac{1274\cdot3564}{(1\cdot04)^2} = £1178.\ 4s.\ 3\tfrac{1}{2}d.$$

$$z = \frac{x}{(1\cdot04)^5} = \frac{1274\cdot3564}{(1\cdot04)^5} = £1047.\ 8s.\ 7d.$$

15. Let P_x = his property at the end of the x^{th} year; therefore the interest in the $(x+1)^{\text{th}}$ year $= rP_x$; and his expenditure in the $(x+1)^{\text{th}}$ year $= (x+1)\,nrP_x$;

$$\therefore \text{the property left} = P_x(1 + r) - (x + 1)\,nrP_x,$$

whence, putting $2t - 1$ for x we have

$$(1 + r - 2tnr)\,P_{2t-1} = 0;$$

$$\therefore 1 + r = 2tnr.$$

Similarly his expenditure in the t^{th} year is $tnrP_{t-1}$, which is also the property he had left at the end of that year.

18. $P = A\dfrac{(1+r)^n - 1}{r(1+r)^n}$ and $P = 250$, $n = 65\frac{1}{2}$, $r = \cdot 07$;

$$\therefore A = \frac{Pr(1+r)^n}{(1+r)^n - 1} = \frac{250 \times \cdot 07 \times (1\cdot 07)^{\frac{131}{2}}}{(1\cdot 07)^{\frac{131}{2}} - 1}$$

$$= \frac{1471\cdot 22}{83\cdot 0696} = 17\cdot 7107,$$

or $A = £17.\ 14s.\ 2\frac{1}{2}d.$ nearly.

20. $$P = \frac{A}{r(1+r)^n}; \quad \therefore Pr(1+r)^n = A,$$

whence $$n = \frac{\log A - \log P - \log r}{\log(1+r)}$$

$$= \frac{\log 85 - \log 946\cdot 63194 - \log \cdot 05}{\log 1\cdot 05}$$

$= 12$ years.

22. Let x the required annuity;

the present value of the 1st annuity $= \left\{\dfrac{(1\cdot 0325)^{25} - 1}{\cdot 0325 \times (1\cdot 0325)^{37}}\right\} \times 50$,

.... 2nd .... $= \left\{\dfrac{(1\cdot 0325)^{25} - 1}{\cdot 0325 \times (1\cdot 0325)^{25}}\right\} \times x$;

$$\therefore \left\{\frac{(1\cdot 0325)^{25} - 1}{\cdot 0325 \times (1\cdot 0325)^{25}}\right\} x = \left\{\frac{(1\cdot 0325)^{25} - 1}{\cdot 0325 \times (1\cdot 0325)^{37}}\right\} \times 50,$$

whence $$x = \frac{50}{(1\cdot 0325)^{12}} = \frac{50}{1\cdot 46785}$$

$= 34\cdot 0635$

$= £34.\ 1s.\ 3\frac{1}{4}d.$

THEORY OF EQUATIONS.

Ex. 1.

5. $$x(x-7)(x+7)\left(x-\frac{1+\sqrt{-1}}{3}\right)\left(x-\frac{1-\sqrt{-1}}{3}\right)=0,$$

$$\text{or } x(x^2-49)\left(x^2-\frac{2x}{3}+\frac{2}{9}\right)=0,$$

$$\text{or } 9x^5-6x^4-439x^3+294x^2-98x=0.$$

9. The coefficient of the 4th term is equal to the product of every three roots with their signs changed

$$\begin{aligned}&=(-2).(-1).(1)+(-2).(-1)(3)+(-2)(-1)(4)+(-2)(+1)(3)\\&\quad+(-2)(1)(4)+(-2)(3)(4)+(-1)(1)(3)+(-1)(1)(4)\\&\quad+(-1)(3)(4)+1.3.4\\&=2+6+8-6-8-24-3-4-12+12=-29;\end{aligned}$$

$$\therefore \text{ the 4th term} = 29x^2.$$

11. Since $7, 6, 4, 2$ are divisors of 336, which $=7.6.4.2$, and $7+6+4+2=19$, the coefficient of the 2nd term;

$\therefore$ $7, 6, 4, 2$ are the roots of the equation.

Ex. 2.

1. The equation is then divisible by $x-4$; to find the quotient.

$$\text{Division}\quad \begin{array}{r|rrrrr} 1 & 1 & -19 & +132 & -302 & +56 \\ +4 & & +4 & -60 & +288 & -56 \\ \hline & 1 & -15 & +72 & -14 & +0 \end{array}$$

$\therefore$ $x^3-15x^2+72x-14=0$ is the required equation.

3. The equation is divisible by

$$(x-2)(x-6), \text{ or } x^2-8x+12;$$

$$\text{Division}\quad \begin{array}{r|rrrrrr} 1 & 1 & -2 & -67 & +200 & +588 & -1440 \\ +8 & & +8 & +48 & -248 & -960 & \\ -12 & & & -12 & -72 & +372 & +1440 \\ \hline & 1 & +6 & -31 & -120 & +0 & +0 \end{array}$$

$\therefore x^3+6x^2-31x-120=0$ is the required equation.

Ex. 8.

3. The coefficients being rational, 3, $+\sqrt{2}$ and $-\sqrt{2}$ must be roots, therefore the equation is divisible by

$$(x-3)(x-\sqrt{2})(x+\sqrt{2}) = x^3-3x^2-2x+6.$$

$$\text{Division}\quad \begin{array}{r|rrrrrr} 1 & 1 & -10 & +29 & -10 & -62 & +60 \\ +3 & & +3 & -21 & +30 & & \\ +2 & & & +2 & -14 & +20 & \\ -6 & & & & -6 & +42 & -60 \\ \hline & 1 & -7 & +10 & +0 & +0 & +0 \end{array}$$

$$\therefore x^2-7x+10=0=(x-2)(x-5);$$

$$\therefore x=2, \text{ and } x=5.$$

4. Let the roots be a, a, b.

Then $a+a+b=7$, and $a(a+b)+ab=16$;

$\therefore a^2+2ab$, or $a^2+2a(7-2a)=16$;

$\therefore 3a^2-14a+16$, or $(3a-8)(a-2)=0$.

Hence the roots are 2, 2, 3.

11. Let the roots be $a-d$, a, and $a+d$,

then their sum $3a=9$; $\therefore a=3$,

also their product $a(a^2-d^2)=15$;

$\therefore 9-d^2=5$, whence $d=\pm 2$;

$\therefore$ the roots are 1, 3, 5.

13. Let $a-3d$, $a-d$, $a+d$, $a+3d$ be the roots, then the sum $4a=10$; the product of every 2 and 2,

$$6a^2-10d^2=35.$$

Hence $a=\frac{5}{2}$, $d=\frac{1}{2}$; $\therefore$ the roots are 1, 2, 3, 4.

15. Let $\frac{a}{r}$, a, ar be the roots,

then $a^3=27$, $a=3$,

and $\frac{a}{r}+a+ar=\frac{3}{r}+3+3r=13$,

or $3r^2-10r=-3$,

whence $r=3$, or $\frac{1}{3}$;

$\therefore$ the roots are 1, 3, 9.

20. Then $a+\frac{1}{a}+b+\frac{1}{b}=\frac{35}{6}$;

and $a\left(\frac{1}{a}+b+\frac{1}{b}\right)+\frac{1}{a}\left(b+\frac{1}{b}\right)+1=\frac{62}{6}$;

$$\therefore \left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)=\frac{62}{6}-2=\frac{50}{6}.$$

Hence $a+\frac{1}{a}-\left(b+\frac{1}{b}\right)=\left\{\left(\frac{35}{6}\right)^2-4\left(\frac{50}{6}\right)\right\}^{\frac{1}{2}}=\frac{5}{6}$;

$$\therefore a+\frac{1}{a}=\frac{10}{3},\quad b+\frac{1}{b}=\frac{5}{2};$$

$\therefore$ the roots are 3, $\frac{1}{3}$, 2, $\frac{1}{2}$.

Transformations.

Ex. 4.

2. Let $y=6x$; then $\left(\frac{y}{6}\right)^3-\frac{7}{3}\left(\frac{y}{6}\right)^2+\frac{11}{36}\left(\frac{y}{6}\right)-\frac{25}{72}=0$;

$$\therefore y^3-14y^2+11y-75=0.$$

11. Let $x-2=y$, or $x=y+2$, and substitute. Then

$$\begin{array}{rr} (y+2)^5 = & y^5+10y^4+40y^3+80y^2+80y+32 \\ 2(y+2)^4 = & 2y^4+16y^3+48y^2+64y+32 \\ -15(y+2)^3 = & -15y^3-90y^2-180y-120 \\ -12(y+2)^2 = & -12y^2-48y-48 \\ -76(y+2) = & -76y-152 \\ -80 = & -80 \\ \hline \end{array}$$

$\therefore$ the reqd. equation is $y^5+12y^4+41y^3+26y^2-160y-336=0$.

11. Another method.

$$\begin{array}{l} 1+2-15-12-76-80 \\ \quad +2+8-14-52-256 \\ \hline 1+4-7-26-128-336 \\ \quad +2+12+10-32 \\ \hline 1+6+5-16-160 \\ \quad +2+16+42 \\ \hline 1+8+21+26 \\ \quad +2+20 \\ \hline 1+10+41 \\ \quad +2 \\ \hline 1+12 \end{array}$$

$\therefore y^5+12y^4+41y^3+26y^2-160y-336=0$ is the transformed equation.

15.

$$\begin{array}{l} 2+0-13+10-19 \\ \quad +2+2-11-1 \\ \hline 2+2-11-1-20 \\ \quad +2+4-7 \\ \hline 2+4-7-8 \\ \quad +2+6 \\ \hline 2+6-1 \\ \quad +2 \\ \hline 2+8 \end{array}$$

$\therefore 2y^4+8y^3-y^2-8y-20=0$ is the transformed equation.

Ex. 5.

1. Transform the equation into one whose roots are those of the original, *diminished* each by $\frac{6}{3}$ or 2; thus

$$\left.\begin{array}{l}
1 - 6 + 7 - 2 \\
\quad + 2 - 8 - 2 \\
\hline
1 - 4 - 1 - 4 \\
\quad + 2 - 4 \\
\hline
1 - 2 - 5 \\
\quad + 2 \\
\hline
1 + 0
\end{array}\right\} \therefore y^3 - 5y - 4 = 0 \text{ is the required equation.}$$

11. The roots of the transformed equation are to be those of the original, *increased* each by $\frac{15}{9}$ or $\frac{5}{3}$; thus

$$\left.\begin{array}{l}
3 + 15 + 25 - 3 \\
\quad - 5 - \frac{50}{3} - \frac{125}{9} \\
\hline
3 + 10 + \frac{25}{3} - \frac{152}{9} \\
\quad - 5 - \frac{25}{3} \\
\hline
3 + 5 + 0 \\
\quad - 5 \\
\hline
3 + 0
\end{array}\right\} \therefore 3y^3 - \frac{152}{9}, \text{ or } 27y^3 - 152 = 0 \text{ is the required equation.}$$

14. To take away the third term, the general formula is

$$\frac{1}{2} n(n-1) mk^2 + (n-1) pk + q = 0.$$

Here $n = 3$, $m = 1$, $p = 6$, $q = 9$; $\therefore k = 3$, or 1.

$$\begin{array}{lr} (1) & 1-6+9-20 \\ & +3-9+\ \ 0 \\ \hline & 1-3+0-20 \\ & +3+0 \\ \hline & 1+0+0 \\ & +3 \\ \hline & 1+3 \end{array} \qquad \begin{array}{lr} (2) & 1-6+9-20 \\ & +1-5+\ \ 4 \\ \hline & 1-5+4-16 \\ & +1-4 \\ \hline & 1-4+0 \\ & +1 \\ \hline & 1-3 \end{array}$$

$$\therefore\ y^3+3y^2-20=0. \qquad \therefore\ y^3-3y^2-16=0.$$

17. Here $\frac{1}{2}n(n-1)mk^2+(n-1)pk+q=0$ gives

$$2\cdot 3\cdot 3k^2-3\cdot 4k+2=0,$$

$$(3k)^2-2(3k)+1=0;\quad \therefore\ k=\frac{1}{3}.$$

The two roots of the equation in k being equal, there is only one transformed equation, which by the process in (14) is found to be $27y^4+67y-59=0$.

19. Let $m=1$, and a, b, c......l, be the n roots.

$$\text{Then } \frac{1}{2}n(n-1)k^2+(n-1)pk+q=0,$$

$$nk+p=0.$$

Eliminating k, $\frac{1}{2}\frac{n-1}{n}p^2-(n-1)\frac{p^2}{n}+q=0;\quad \therefore\ \frac{p^2}{2q}=\frac{n}{n-1};$

$$\therefore\ p^2 : p^2-2q=n : 1;$$

i.e. $(a+b+c+\ldots\ldots+l)^2 : (a^2+b^2+c^2+\ldots\ldots+l^2)=n : 1.$

SYMMETRICAL FUNCTIONS.

Ex. 6.

1. Comparing $x^3 - x - 1 = 0$, with $x^3 + px^2 + qx + r = 0$; we have $p = 0$, $q = -1$, $r = -1$.

$$\begin{aligned}
\text{Then } S_1 + p &= 0; & \therefore S_1 &= 0.\\
S_2 + pS_1 + 2q &= 0; & \therefore S_2 &= +2.\\
S_3 + pS_2 + qS_1 + 3r &= 0; & \therefore S_3 &= +3.\\
S_4 + pS_3 + qS_2 + rS_1 &= 0; & \therefore S_4 &= +2.\\
S_5 + pS_4 + qS_3 + rS_2 &= 0; & \therefore S_5 &= +5.\\
S_6 + pS_5 + qS_4 + rS_3 &= 0; & \therefore S_6 &= +5.
\end{aligned}$$

2. To find S_{-2}; put $x = \frac{1}{y}$, and in the transformed equation find S_2.

Ex. 7.

1. Let the transformed equation be

$$y^3 - Qy + R = 0,$$

then if a, b, c be the roots of the proposed equation,

$$\begin{aligned}
Q &= (a+b)(a+c) + (a+b)(b+c) + (a+c)(b+c)\\
&= a^2 + b^2 + c^2 + 3(ab + ac + bc)\\
&= (a+b+c)^2 + (ab + ac + bc)\\
&= (ab + ac + bc), \text{ since the 2nd term of the equation is wanting,}\\
&= 40.
\end{aligned}$$

$$\begin{aligned}
\text{Similarly } R &= (a+b)(a+c)(b+c)\\
&= (a+b+c)(ab+ac+bc) - r\\
&= -r = -39.
\end{aligned}$$

Therefore the transformed equation is

$$x^3 - 40x - 39 = 0.$$

Ex. 8.

2. Let $x^3 - Px^2 + Qx - R = 0$ be the transformed equation, then (1) $P = ab + ac + bc = q$,

$$Q = ab \,.\, ac + ab \,.\, bc + ac \,.\, bc = a(abc) + b\,(abc) + c\,(abc)$$
$$= abc\,(a + b + c) = pr,$$
$$R = ab \,.\, ac \,.\, bc = a^2b^2c^2 = r^2;$$

therefore the transformed equation is

$$x^3 - qx^2 + prx - r^2 = 0.$$

4. $P = a^2 + b^2 + a^2 + c^2 + b^2 + c^2 = 2\,(a^2 + b^2 + c^2)$

$$= 2S_2 = 2\,(p^2 - 2q),$$
$$Q = (a^2 + b^2)(a^2 + c^2) + (a^2 + b^2)\,(b^2 + c^2) + (a^2 + c^2)(b^2 + c^2)$$
$$= (a^4 + b^4 + c^4) + 3\,(a^2b^2 + a^2c^2 + b^2c^2)$$
$$= S_4 + \frac{3}{2}\,\{(S_2)^2 - S_4\} = \frac{3\,(S_2)^2 - S_4}{2}$$
$$= \frac{3\,(p^4 - 4qp^2 + 4q^2) - (p^4 - 4qp^2 + 2q^2 + 4rp)}{2}$$
$$= p^4 - 4p^2q + 5q^2 - 2pr,$$
$$R = (a^2+b^2)(a^2+c^2)(b^2+c^2) = a^4b^2 + a^2b^4 + a^4c^2 + a^2c^4 + 2a^2b^2c^2$$
$$= S_4 \,.\, S_2 - S_6 + 2r^2$$
$$= -\,2rp^3 + q^2p^2 + 4qrp - 2q^3 - 3r^2 + 2r^2$$
$$= p^2q^2 - 2p^3r + 4pqr - 2q^3 - r^2;$$

therefore the transformed equation is

$$x^3 - 2\,(p^2 - 2q)\,x^2 + (p^4 - 4p^2q + 5q^2 - 2pr)\,x$$
$$-\,(p^2q^2 - 2p^3r + 4pqr - 2q^3 - r^2) = 0.$$

7. $P = a^3 + b^3 + c^3 = S_3 = p^3 - 3pq + 3r,$

$$Q = a^3b^3 + a^3c^3 + b^3c^3 = \frac{(S_3)^2 - S_6}{2}$$
$$= q^3 + 3r^2 - 3pqr,$$
$$R = a^3b^3c^3 = r^3;$$

substituting these values in the equation

$$x^3 - Px^2 + Qx - r = 0,$$

we have for the transformed equation

$$x^3 - (p^3 - 3pq + 3r)\,x^2 + (q^3 + 3r^2 - 3pqr)\,x - r^3 = 0.$$

Ex. 9.

1. Dividing the equation by x^2, we have

$$x^2 - 10x + 26 - \frac{10}{x} + \frac{1}{x^2} = 0,$$

$$\text{or } \left(x + \frac{1}{x}\right)^2 - 10\left(x + \frac{1}{x}\right) + 24 = 0,$$

$$\text{whence } x + \frac{1}{x} = 4, \text{ or } 6;$$

$$\therefore x^2 - 4x = -1, \text{ or } x^2 - 6x = -1,$$

$$\text{whence } x = 2 \pm \sqrt{3}, \quad \text{whence } x = 3 \pm 2\sqrt{2};$$

therefore the roots are $3 \pm 2\sqrt{2}$, $2 \pm \sqrt{3}$.

6. 1 is evidently a root. Therefore, dividing the equation by $x - 1$, we have

$$x^4 - \frac{13}{2}x^3 + 12x^2 - \frac{13}{2}x + 1 = 0,$$

$$\text{whence } \left(x + \frac{1}{x}\right)^2 - \frac{13}{2}\left(x + \frac{1}{x}\right) + 10 = 0,$$

$$\text{whence } x + \frac{1}{x} = 4, \text{ or } \frac{5}{2};$$

therefore the roots are $1, 2, \frac{1}{2}, 2 \pm \sqrt{3}$.

10. Here -1 is a root; therefore dividing by $(x + 1)$ we get

$$x^8 - 9x^6 + 12x^5 - 20x^4 + 12x^3 - 9x^2 + 1 = 0;$$

$$\therefore \left(x^4 + \frac{1}{x^4}\right) - 9\left(x^2 + \frac{1}{x^2}\right) + 12\left(x + \frac{1}{x}\right) - 20 = 0.$$

$$\text{Assume } x + \frac{1}{x} = y,$$

$$\text{then } y^4 - 4y^2 + 2 - 9y^2 + 18 + 12y - 20 = 0,$$

$$\text{or } y^4 - 13y^2 + 12y = 0;$$

$$\therefore y(y - 1)(y - 3)(y + 4) = 0.$$

Putting $x+\frac{1}{x}=0,\ 1,\ 3,\ -4$, successively; we obtain all the roots of the original,

$$-1,\ \pm\sqrt{-1},\ \ \frac{1}{2}\{1\pm\sqrt{-3}\},\ \ \frac{1}{2}(3\pm\sqrt{5}),\ \ -2\pm\sqrt{3}.$$

14. $$x^4+1=0, \qquad x^4-1=0;$$

$$x^2+\frac{1}{x^2}=0; \qquad \therefore\ x^2-\frac{1}{x^2}=\left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)=0,$$

$$\therefore\ x+\frac{1}{x}=\pm\sqrt{2}, \qquad x-\frac{1}{x}=0;\quad \therefore\ x=\pm 1.$$

Similarly $x-\frac{1}{x}=\pm\sqrt{-2}$; $\qquad x+\frac{1}{x}=0$; $\quad\therefore\ x=\pm\sqrt{-1}$;

$$\therefore\ x=\frac{1}{2}\{\pm\sqrt{2}\pm\sqrt{-2}\}.$$

14_2. $x^9=-1=\cos(2\lambda+1)\pi\pm\sqrt{-1}\sin(2\lambda+1)\pi$ in its most general form.

By Demoivre's Theorem,

$$x=\cos\frac{(2\lambda+1)\pi}{9}\pm\sqrt{-1}\sin\frac{(2\lambda+1)\pi}{9}.$$

Hence, by giving to λ the values 0, 1, 2, 3, 4 successively, the roots are found to be -1, $\cos(n+20)\pm\sqrt{-1}\sin(n+20)$, where n is 1, 3, 5, 7 successively.

Ex. 10.

1. $X=x^3-2x^2-15x+36=0$;

$\therefore\ X_1=3x^2-4x-15=(x-3)(3x+5)=0$, the 1st derived.

Hence $x-3$ is the Greatest Common Measure of X and X_1;

$$\therefore\ (x-3)^2 \text{ is a factor of } X.$$

By division, it appears that $X=(x-3)^2(x+4)$;

$$\therefore\ \text{the roots are } 3,\ 3,\ -4.$$

5. $$X = 2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0,$$
$$X_1 = 8x^3 - 36x^2 + 38x - 6 = 0.$$

The G. C. M. of X and X_1 is $x - 3$;

$$\therefore\ X = (x-3)^2 (2x^2 + 1) = 0;$$

$$\therefore \text{ the roots are } 3,\ 3,\ \pm\left(-\frac{1}{2}\right)^{\frac{1}{2}}.$$

12. The Greatest Common Measure of X and X_1 is
$$x^4 - 7x^3 + 13x^2 + 3x - 18 = 0 = D \text{ suppose},$$
then $4x^3 - 21x^2 + 26x + 3 = 0 = D_1$ the 1st derived of D.

The G. C. M. of D and D_1 is $x - 3$;

$$\therefore\ (x-3)^2 \text{ is a factor of } D.$$

Hence $D = (x-3)^2 (x^2 - x - 2) = (x-3)^2 (x-2)(x+1)$.

Similarly $(x-3)^3 (x-2)^2 (x+1)^2$ is a factor of X;

$$\therefore\ X = (x-3)^3 (x-2)^2 (x+1)^2 (x+4);$$

$$\therefore \text{ the roots are } 3,\ 3,\ 3,\ 2,\ 2,\ -1,\ -1,\ -4.$$

13. Let $X = Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots + Kx + L = 0$,

and $$Y = A\alpha x^n + B(\alpha + \beta) x^{n-1} + C(\alpha + 2\beta) x^{n-2} + \ldots$$
$$+ K\{\alpha + (n-1) . \beta\} x + L(\alpha + n\beta)$$
$$= \alpha (Ax^n + Bx^{n-1} + \ldots + Kx + L)$$
$$+ \beta \{Bx^{n-1} + 2Cx^{n-2} + \ldots + (n-1) Kx + nL\}.$$

Now $X_1 = nAx^{n-1} + (n-1) Bx^{n-2} + (n-2) Cx^{n-3} + \ldots + K = 0$,

since $X = 0$ has two equal roots,

$$= n\left(Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \ldots + K + \frac{L}{x}\right)$$
$$- \left\{Bx^{n-2} + 2Cx^{n-3} + \ldots + (n-1) K + \frac{nL}{x}\right\};$$

$$\therefore\ Bx^{n-1} + 2Cx^{n-2} + \ldots + (n-1) Kx + nL = nX - xX_1.$$

Hence $Y = \alpha X + \beta (nX - xX_1) = 0$.

15. $X = x^5 + qx^3 - rx^2 - t = 0$, has two equal roots;

$\therefore X' = 5x^4 + 3qx^2 - 2rx = 0$.

Proceed as for finding the G. C. M. of X and X'.

Then $5X = \left(x^2 + \frac{2}{5}q\right)X_1 - \left(3rx^2 + \frac{6}{5}q^2x + 5t - \frac{4qr}{5}\right)$.

Hence, a value of x which makes $X = 0$, and $X' = 0$, must also make

$$3rx^2 + \frac{6}{5}q^2x + 5t - \frac{4qr}{5} = 0;$$

therefore one of the equal roots is a root of

$$x^2 + \frac{2q^2}{5r}x + \frac{5t}{3r} - \frac{4q}{15} = 0.$$

Ex. 11.

2. Let $x = \frac{1}{y}$; then $1 - 11y + 36y^2 - 36y^3 = 0$;

$$\therefore y^3 - y^2 + \frac{11}{36}y - \frac{1}{36} = 0.$$

Since the roots of the original are in Harm. Prog., the roots of the transformed are in Arith. Prog.

Let $a - h$, a, $a + h$ be the roots of the transformed.

$$\text{Then } 3a = 1 \text{ and } a(a^2 - h^2) = \frac{1}{36};$$

$\therefore a = \frac{1}{3}$, $h = \frac{1}{6}$; $\therefore \frac{1}{6}, \frac{1}{3}, \frac{1}{2}$ are the values of y;

$\therefore$ 2, 3, 6 are the roots of the original equation.

9. Let $\alpha, \beta, \gamma, \delta$ be the roots of the proposed equation.

$$\text{If } \alpha\beta = 30, \text{ then } \gamma\delta = \frac{1200}{30} = 40,$$

$$\alpha + \beta + \gamma + \delta = -1,$$

$$\alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma\delta = -62;$$

$$\therefore (\alpha + \beta)(\gamma + \delta) = -62 - 30 - 40 = -132.$$

Hence

$$\alpha+\beta-(\gamma+\delta)=\sqrt{1+4\times 132}=\sqrt{529}=23;$$

$$\therefore \alpha+\beta=11; \quad \gamma+\delta=-12;$$

$$\alpha-\beta=\sqrt{11^2-4\times 30}=1; \quad \gamma-\delta=\sqrt{12^2-4\times 40}=\pm 4\sqrt{-1};$$

$$\therefore \alpha=5, \ \beta=6, \ \gamma \text{ and } \delta=-6\pm 2\sqrt{-1}.$$

Ex. 12.

1. Since $\left(-\frac{14}{2}\right)^2+\left(-\frac{9}{3}\right)^3$ is positive, Cardan's Method is applicable.

Let $x=y+z$; then $y^3+z^3+(3yz-9)(y+z)-14=0$.
Also let $3yz-9=0$, then $y^3+z^3=14$;

$$\therefore y^3-z^3=\sqrt{14^2-4\times 3^3}=\pm 2\sqrt{22}.$$

$$\text{Hence } y^3=7\pm\sqrt{22}, \quad z^3=7\mp\sqrt{22};$$

$$\therefore x=(11\cdot 6904158)^{\frac{1}{3}}+(2\cdot 3095842)^{\frac{1}{3}}=3\cdot 591.$$

11. Assume $y+\frac{8}{3}=x$, then $y^3-\frac{82}{3}y-\frac{1213}{27}=0$.

Since $\left(\frac{1213}{54}\right)^2-\left(\frac{82}{9}\right)^3$ is negative, Cardan's method is not applicable.

$$\text{Let } y=m\cos\theta,$$

$$\text{then } m^3\cos^3\theta-\frac{82m}{3}\cos\theta-\frac{1213}{27}=0,$$

$$\text{or } \cos^3\theta-\frac{82}{3m^2}\cos\theta-\frac{1213}{27m^3}=0.$$

$$\text{Now } \cos^3\theta-\frac{3}{4}\cos\theta-\frac{1}{4}\cos 3\theta=0;$$

$$\therefore \frac{82}{3m^2}=\frac{3}{4}, \quad \therefore m=\frac{2}{3}\sqrt{82},$$

$$\text{and } \frac{1}{4}\cos 3\theta=\frac{1213}{27m^3}; \quad \therefore \cos 3\theta=\frac{1213\sqrt{82}}{13448}.$$

Hence, by the tables,

$$3\theta = 35^\circ\ 14'\ 7''\cdot187;\ \text{and}\ \therefore\ \theta = 11^\circ\ 44'\ 42''\cdot396;$$

$$\therefore\ y = \frac{2}{3}\sqrt{82}\cos\theta = 5\cdot9105278.$$

Also $y = \frac{2}{3}\sqrt{82}\cos\left(\frac{2\pi+3\theta}{3}\right)$, or $\frac{2}{3}\sqrt{82}\cos\left(\frac{2\pi-3\theta}{3}\right)$

$$= -\frac{2}{3}\sqrt{82}\cos(48^\circ 15' 17''\cdot603),$$

$$\text{or}\ -\frac{2}{3}\sqrt{82}\cos(71^\circ\ 44'\ 42''\cdot396)$$

$$= -4\cdot0194923,\ \text{or}\ -1\cdot8910355;$$

$$\therefore\ x = 8\cdot5771945,\ -1\cdot35282507,\ \cdot77563.$$

BIQUADRATIC EQUATIONS.

Ex. 13.

1. By Des Cartes' Method.

Let $(x^2+ex+f)(x^2-ex+g) = x^4-3x^2-42x-40=0.$

Equating the coefficients of the same powers of x, we get

$$g+f-e^2=-3,\quad eg-ef=-42,\quad fg=-40.$$

Then

$$2g\times2f = \left(e^2-3-\frac{42}{e}\right)\left(e^2-3+\frac{42}{e}\right) = e^4-6e^2+9-\frac{1764}{e^2} = -4\times40;$$

$$\therefore\ e^6-6e^4+169e^2-1764=0.$$

Let $e^2=y$, then $y^3-6y^2+169y-1764=0.$

By trial of square numbers, 9 is found to be a root of this cubic;

$$\therefore\ e=3,\ f=\frac{1}{2}(9-3+14)=10,\ g=-4.$$

Hence $x^2+3x+10=0$; $\therefore\ x=\frac{1}{2}\{-3\pm\sqrt{-31}\}$,

and $x^2+3x-\ 4=0$; $\therefore\ x=4$, and -1.

2. $$x^4 - 25x^2 - 60x - 36 = 0.$$

By Euler's Method.

$$\text{Let } x = y + z + u.$$

$$\text{Then } x^2 = y^2 + z^2 + u^2 + 2(yz + yu + zu);$$

$$\therefore x^4 - 2x^2(y^2 + z^2 + u^2) + (y^2 + z^2 + u^2)^2$$
$$= 4(y^2z^2 + y^2u^2 + z^2u^2) + 8yzu(y + z + u),$$

or $x^4 - 2x^2(y^2 + z^2 + u^2) - 8xyzu + (y^2 + z^2 + u^2)^2$
$$- 4(y^2z^2 + y^2u^2 + z^2u^2) = 0.$$

Comparing this equation with the proposed

$x^4 - 25x^2 - 60x - 36 = 0$, we have

$$y^2 + z^2 + u^2 = \frac{25}{2}, \quad yzu = \frac{60}{8} = \frac{15}{2},$$

$$(y^2 + z^2 + u^2)^2 - 4(y^2z^2 + y^2u^2 + z^2u^2) = -36;$$

$$\therefore y^2z^2 + y^2u^2 + z^2u^2 = 9 + \frac{1}{4}\left(\frac{25}{2}\right)^2 = \frac{769}{16}.$$

Hence the values of y^2, z^2, u^2 are the roots of the equation

$$t^3 - \frac{25}{2}t^2 + \frac{769}{16}t - \frac{225}{4} = 0.$$

Let $t = \frac{k}{4}$; then $k^3 - 50k^2 + 769k - 3600 = 0$, the roots of which are 9, 16, 25; $\therefore$ the values of t are $\frac{9}{4}$, $\frac{16}{4}$, $\frac{25}{4}$.

$$\text{Hence } y = \pm\frac{3}{2}, \quad z = \pm 2, \quad u = \pm\frac{5}{2}.$$

Now $x = y + z + u$, with the condition that $yzu = +\frac{15}{2}$;

$$\therefore x = +\frac{3}{2} + 2 + \frac{5}{2} = +6,$$

$$x = -\frac{3}{2} - 2 + \frac{5}{2} = -1,$$

$$x = -\frac{3}{2} + 2 - \frac{5}{2} = -2,$$

$$x = +\frac{3}{2} - 2 - \frac{5}{2} = -3.$$

4. $$x^4 - 4x^3 - 8x + 32 = 0.$$

By Ferrari's Method.

Take away the second term by putting $x = y + 1$, the result is

$$y^4 - 6y^2 - 16y + 21 = 0.$$

$$\text{Then } y^4 = 6y^2 + 16y - 21,$$

add $2ky^2 + k^2$ to both sides;

$$\therefore\ (y^2 + k)^2 = (2k + 6)y^2 + 16y + k^2 - 21 \text{ a perfect square};$$

$$\therefore\ (2k + 6)(k^2 - 21) = \frac{16^2}{4} = 64;$$

$$\therefore\ k^3 + 3k^2 - 21k - 95 = 0; \text{ of which } 5 \text{ is a root.}$$

$$\text{Hence } y^2 + 5 = \pm(4y + 2);$$

$$\therefore\ y^2 - 4y + 3 = 0; \quad \therefore\ y = 1, \text{ and } y = 3,$$

$$y^2 + 4y + 7 = 0; \quad \therefore\ y = -2 \pm \sqrt{-3}.$$

The roots of the original are $\therefore\ 4, 2, -1 \pm \sqrt{-3}$.

Limits of Roots.

Ex. 14.

1. Transform the equation into one whose roots shall be less by h than the roots of the given equation,

$$\text{then } (y + h)^3 - 4(y + h)^2 - 4(y + h) + 20 = 0,$$

or $(h^3 - 4h^2 - 4h + 20) + (3h^2 - 8h - 4)y + (3h - 4)y^2 + y^3 = 0 \ldots (1)$.

Substitute the numbers 1, 2, 3, 4, &c. successively for h in (1), and the first number which makes *all* the coefficients positive will be found to be 4;

$\therefore$ 4 is the No. next greater than the greatest positive root.

4.
$$\begin{aligned}
fx &= x^4 - 2x^3 - 3x^2 - 15x - 3 = 0,\\
f'x &= 4x^3 - 6x^2 - 6x - 15 = 0,\\
f''x &= 12x^2 = 12x - 6 = 0,\\
f'''x &= 24x - 12 = 0,\\
f^{IV}x &= 24
\end{aligned}$$

And substituting the numbers 1, 2, 3, 4, &c. successively for x in these expressions, we find that 4 is the first number that makes them all positive;

$\therefore$ 4 is the superior limit to the roots.

10. There are but *two* changes of sign, viz. from + to −, and from − to +, therefore there cannot be more than 2 positive roots.

Again, there is but *one* permanence of the same sign, viz. −, −, therefore the equation cannot have more than 1 negative root.

RATIONAL ROOTS.

Ex. 15.

1. Here the limits of the roots are 7 and −1, and the divisors of the last term are therefore 6, 4, 3, 2;

$$\therefore a = \quad 6, \quad 4, \quad 3, \quad 2,$$

$$q_1 = -\frac{24}{a} = -4, \quad -6, \quad -8, \quad -12,$$

$$q_1 + 22 = \quad 18, \quad 16, \quad 14, \quad 10,$$

$$q_2 = \frac{q_1+22}{a} = \quad 3, \quad 4, \quad \times, \quad 5,$$

$$q_2 + (-9) = -6, \quad -5, \quad \times, \quad -4,$$

$$q_3 = \frac{q_2 - 9}{a} = -1, \quad \times, \quad \times, \quad -2;$$

therefore 6 is the only commensurable root, since it does not satisfy the equation $3x^2 - 18x + 22 = 0$; $\therefore$ dividing the given equation by $x - 6$, we have

$$x^2 - 3x + 4 = 0,$$

$$\text{whence } x = \frac{1}{2}\{3 \pm (-7)^{\frac{1}{2}}\}.$$

4. $\qquad x^3 - 5x^2 - 18x + 72 = 0 = f(x)$ suppose.

Changing the signs of the alternate terms, we have

$$x^3 + 5x^2 - 18x - 72 = 0.$$

Hence the roots of $f(x) = 0$ lie between 7 and -5.

$$\text{But } f(1) = 50, \quad f(-1) = 84;$$

$\therefore$ the only admissible divisors of 72 are those which when diminished by 1, divide 50 without remainder, and are

$$6, \ 3, \ 2, \ -4;$$

also the only admissible divisors increased by 1 which divide 84, are

$$6, \ 3, \ -2, \ -3;$$

$\therefore$ 6, 3, 2, -2, -3, -4 are the only divisors which need be tried;

$$\begin{array}{rrrrrrr} \therefore \ a = & 6, & 3, & 2, & -2, & -3, & -4, \\ q_1 = & 12, & 24, & 36, & -36, & -24, & -18, \\ q_1 + (-18) = & -6, & 6, & 18, & -54, & -42, & -36, \\ q_2 = & -1, & 2, & 9, & 27, & 14, & 9, \\ q_2 + (-5) = & -6, & -3, & 4, & 22, & 9, & 4, \\ q_3 = & -1, & -1, & 2, & -11, & -3, & -1; \end{array}$$

$\therefore$ the roots are 6, 3, -4.

9. $$8x^3 - 26x^2 + 11x + 10 = 0.$$

Let $y = 8x$, then $y^3 - 26y^2 + 88y + 640 = 0 = f(y)$, the roots of which lie between 27 and -7.

$$\text{But } f(1) = 703 \text{ and } f(-1) = 535;$$

$\therefore$ the only admissible divisors of 640, which, when diminished by 1, divide 703, are 20, 2.

And the only admissible divisors increased by 1 which divide 535 are 4, -2;

$\therefore$ the only divisors which need be tried are 20, 4, ± 2;

$$\begin{array}{rrrrr} \therefore \ a = & 20, & 4, & 2, & -2, \\ q_1 = & 32, & 160, & 320, & -320, \\ q_1 + 88 = & 120, & 248, & 408, & -232, \\ q_2 = & 6, & 62, & 204, & 116, \\ q_2 - 26 = & -20, & 36, & 178, & 90, \\ q_3 = & -1, & 9, & 89, & -45; \end{array}$$

$\therefore$ 20 is the only commensurable root, for it is not a root of the equation $3y^2 - 52y + 88 = 0$;

$\therefore \frac{20}{8} = \frac{5}{2}$ is the only commensurable root of the given equation. Divide the given equation by $2x - 5$, and we have

$$4x^2 - 3x - 2 = 0; \text{ whence } x = \frac{1}{8}(3 \pm \sqrt{41}).$$

12. Assume $x = \frac{y}{6}$, and we have

$$y^4 + 53y^3 - 570y^2 - 900y + 9072 = 0 = f(y).$$

Let	Results.	Divisors.	Progressions.	
$y = 1$...	7656	1, 2, 3, 4, 6, 8, 11, 12, 22, &c.	3	8
$= 0$...	9072	1, 2, 3, 4, 6, 7, 8, 9, 12, 14, 16, 18, 21, 24, &c.	4	9
$= -1$...	9350	1, 2, 5, 10, 17, 20, &c.	5	10

$\therefore$ the only numbers to be tried are 4, 9, both of which are roots of $f(y) = 0$;

$\therefore x = \frac{y}{6} = \frac{2}{3}$ and $\frac{3}{2}$ the only commensurable roots.

Now divide the given equation by $(3x - 2)(2x - 3) = 0$, and we have

$$x^2 + 11x + 7 = 0,$$

$$\text{whence } x = \frac{1}{2}\{-11 \pm (93)^{\frac{1}{2}}\}.$$

Sturm's Theorem.

Ex. 16.

1. Here $X = x^3 + 2x^2 - 3x + 2$,

$$X_1 = 3x^2 + 4x - 3,)\, 3x^3 + 6x^2 - 9x + 6 \,(\, x + 1$$

$$\underline{3x^3 + 4x^2 - 3x}$$

$$2x^2 - 6x + 6, \quad \times \text{ by } \tfrac{3}{2},$$

$$\text{or } 3x^2 - 9x + 9$$

$$\underline{3x^2 + 4x - 3}$$

$$-13x + 12;$$

$$\therefore X_2 = 13x - 12.$$

$$\begin{array}{r} 3x^2+4x-3 \\ 13 \\ \hline 13x-12\,)\,39x^2+52x-39\,(\,3x+88 \\ 39x^2-36x \\ \hline 88x-39 \\ \hline 1144x-507 \\ 1144x-1056 \\ \hline +549 \end{array}$$

Otherwise,

since $X_1 = X_2 Q_2 - X_3$

if $x = \frac{12}{13}$, then $X_2 = 0$,

and X_1 is positive;

$\therefore$ X_3 is negative.

$$\begin{aligned} \therefore\ X &= x^3 + 2x^2 - 3x + 2, \\ X_1 &= 3x^2 + 4x - 3, \\ X_2 &= 13x - 12, \\ X_3 &= -. \end{aligned}$$

Since there is one change of signs in the 1st terms of X, X_1, X_2, X_3; therefore there is one pair of imaginary roots in $X = 0$.

The limits of the real roots of $X = 0$ are 4 and -4.

	X	X_1	X_2	X_3	
If $x = \infty$, the signs are	+	+	+	$-$,	one change
$x = 0$, ...	+	$-$	$-$	$-$,	....
$x = -1$, ...	+	$-$	$-$	$-$,	....
$x = -2$, ...	+	+	$-$	$-$,	....
$x = -3$, ...	+	+	$-$	$-$,	....
$x = -4$, ...	$-$	+	$-$	$-$,	two changes.

Hence, the real root lies between -3 and -4.

5. Here $X = x^3 - 27x - 36$,

$$\begin{aligned} X_1 &= x^2 - 9, \\ X_2 &= x + 2, \\ X_3 &= +. \end{aligned}$$

Hence all the roots of $X = 0$ are real and lie between the limits $+6$ and -5.

	X	X_1	X_2	X_3	
If $x=\infty$, the signs are	+	+	+	+	∴ one root is positive, and two roots are negative.
$x=0$, ...	−	−	+	+	
$x=-\infty$, ...	−	+	−	+	
$x=6$, ...	+	+	+	+	
$x=5$, ...	−	+	+	+	
$x=-1$, ...	−	−	+	+	
$x=-2$, ...	+	−	0	+	
$x=-3$, ...	+	0	−	+	
$x=-4$, ...	+	+	−	+	
$x=-5$, ...	−	+	−	+	

Hence the positions of the roots are indicated by

$$\{6, 5\}; \quad \{-1, -2\}; \quad \{-4, -5\}.$$

14. Here the functions are

$$X = x^4 - x^3 - 4x^2 + 4x + 1,$$
$$X_1 = 4x^3 - 3x^2 - 8x + 4,$$
$$X_2 = -7x^2 + 8x + 4,$$
$$X_3 = 4x - 5,$$
$$X_4 = +.$$

Since there are two changes in the signs of the 1st terms of X and its four auxiliary functions; ∴ $X=0$ has two pairs of imaginary roots.

15. The functions are

$$X = x^5 - 3x^4 - 24x^3 + 95x^2 - 46x - 101,$$
$$X_1 = 5x^4 - 12x^3 - 72x^2 + 190x - 46,$$
$$X_2 = 276x^3 - 1209x^2 + 350x + 2663,$$
$$X_3 = 887825x^2 - 41054x + 3594025.$$

Since in X_3, four times the product of the extremes is > the square of the mean, therefore every possible value of x will make X_3 positive.

Now $X_3 = X_4 Q_4 - X_5$; therefore the sign of X_5 is always negative; hence whatever be the value of x, the signs of X_3, X_4, X_5 must be $+ \pm -$.

Therefore $X=0$ has one pair of imaginary roots, and the position of the real roots will be indicated by the changes of sign of the functions given above.

		X	X_1	X_2	X_3	
If $x=\infty$,	the signs are	$+$	$+$	$+$	$+$	$\therefore$ one root is $+$,
$x=0$,	....	$-$	$-$	$+$	$+$	two roots are $-$.
$x=-\infty$,	....	$-$	$+$	$-$	$+$	

The limits of the roots of $X=0$, are $+5$ and -6.

If $x=5$,	the signs are	$+$	$+$	$+$	$+$
$x=4$,	....	$-$	$+$	$+$	$+$
$x=0$,	....	$-$	$-$	$+$	$+$
$x=-1$,	....	$+$	$-$	$+$	$+$
$x=-3$,	....	$+$	$-$	$-$	$+$
$x=-5$,	....	$+$	$+$	$-$	$+$
$x=-6$,	....	$-$	$+$	$-$	$+$

Hence $\{5, 4\}$, $\{0, -1\}$, $\{-5, -6\}$.

19. Here the functions are

$$X = x^7 - 2x^5 - 3x^3 + 4x^2 - 5x + 6,$$
$$X_1 = 7x^6 - 10x^4 - 9x^2 + 8x - 5,$$
$$X_2 = 2x^5 + 6x^3 - 10x^2 + 15x - 21,$$
$$X_3 = 62x^4 - 70x^3 + 123x^2 - 163x + 10,$$
$$X_4 = -629x^3 + 1266x^2 - 2830x + 2933,$$
$$X_5 = -18071605x^2 - 476358809x + 105033456,$$
$$X_6 = -46353x + 65080,$$
$$X_7 = -;$$

$\therefore$ if $x=+\infty$, signs $++++----$ give 1 variation,

$x=-\infty$, ... $-+-++-+-$ ... 6 .... ,

the proposed equation has 5 real roots, and one pair of imaginary roots.

The limits of the roots of $X=0$ are $+2$ and -5. If

	Vars.		Vars.
$x=0$, signs $+--++++-$	3	$x=0$, signs $+--++++-$	3
$x=1$, ... $+---+++-$	3	$x=-1$, ... $+--++++-$	3
$x=2$, ... $++++----$	1	$x=-2$, ... $-+-++++-$	4
		$x=-3$, ... $-+-++++-$	4
		$x=-4$, ... $-+-++++-$	4
		$x=-5$, ... $-+-++-+-$	6

hence it appears that the roots lie thus

$$\{2, 1\}, \quad \{2, 1\}, \quad \{-5, -4\}, \quad \{-5, -4\}, \quad \{-2, -1\}.$$

20. Here

$$X = x^6 + 24x^5 + 125x^4 - 376x^3 - 1726x^2 + 5592x - 4080,$$
$$X_1 = 3x^5 + 60x^4 + 250x^3 - 564x^2 - 1726x + 2796,$$
$$X_2 = 5x^4 + 68x^3 + 52x^2 - 908x + 1018{\cdot}4,$$
$$X_3 = 529x^3 + 16356x^2 - 29647x + 13934,$$
$$X_4 = -13159441x^2 + 257661116x - 14426791.$$

If $X_4 = -ax^2 + bx - c$, it is obvious that $(-2a)(-2c)$ is $> b^2$, hence the roots of $X_4 = 0$ are imaginary, and every possible value of x, will make X_4 negative.

Now $X_4 = X_5 Q_5 - X_6$; a value of x which makes $X_5 = 0$, will make x_4 negative and hence X_6, the absolute function is positive.

Hence the signs of X_4, X_5, X_6 will always be $- \pm +$;

$\therefore X = 0$, has two pairs of imaginary roots;

the limits of its real roots are $+2$ and -6,

	X	X_1	X_2	X_3	X_4.	
If $x=\infty$, the signs are	+	+	+	+	−	
$x=2$, ...	+	+	+	+	−	one root
$x=1$, ...	−	+	+	+	−	
$x=0$, ...	−	+	+	+	−	
$x=-2$, ...	−	+	+	+	−	
$x=-4$, ...	−	−	+	+	−	
$x=-5$, ...	−	−	+	+	−	one root
$x=-6$, ...	+	−	+	+	−	

therefore the positions of the two real roots are expressed by $\{2, 1\}$, $\{-5, -6\}$.

APPROXIMATION.

Ex. 17.

2. To find a real root of $x^3-5x-3=3$; by Newton's Method.

Let $X=x^3-5x-3$, and $X'=3x^2-5$.

$$\left.\begin{array}{l} \text{If } x=\ \ 2, \text{ then } X=-\ 5 \\ \quad x=\ \ 3, \ \ldots\ldots\ X=+\ 9 \\ \quad x=2{\cdot}5, \ \ldots\ldots\ X={\cdot}125 \end{array}\right\} \begin{array}{l}\therefore \text{ the real root of } X=0 \text{ lies} \\ \text{between 2 and } 2{\cdot}5, \text{ but nearer to} \\ \text{the latter.}\end{array}$$

Let $x=2{\cdot}5+\delta$.

$$\text{Then } \delta'=-\left(\frac{X}{X'}\right)_{x=2\cdot5}=-\frac{{\cdot}125}{13{\cdot}75}=-\frac{1}{110}=-{\cdot}00909;$$

$\therefore\ x=2{\cdot}4909$, a first approximation.

Again, let $x=2{\cdot}49+\delta'$.

$$\text{Then } \delta'=-\left(\frac{X}{X'}\right)_{x=2\cdot49}=+\frac{{\cdot}011751}{13{\cdot}6}=+{\cdot}00086;$$

$\therefore\ x=2{\cdot}49086$, a second approximation.

6. A root of $6x^3-141x+263=0$, is found to lie between -5 and -6.

Let $x=-y$, then $6y^3-141y=263$.

5 ...	y ...	6
750 ...	$6y^3$ ...	1296
-705 ...	$-141y$ ...	-846
45 ...	$+263$ ...	450

Then $450-45:263-45=6-5:\delta$ the 1st correction;

$\therefore\ \delta=\dfrac{218}{405}={\cdot}53\ldots$ and $y=5{\cdot}53$ is a 1st approximation.

Again,

$5{\cdot}59$ ...	y ...	$5{\cdot}60$
$1048{\cdot}061$ ...	$6y^3$ ...	$1053{\cdot}696$
$-788{\cdot}19$ ...	$-141y$ ...	$-789{\cdot}6$
$259{\cdot}871$ ...	$+263$ ...	$264{\cdot}096$

As before, $4{\cdot}225 : 3{\cdot}129 = {\cdot}01 : \delta'$ the 2nd correction;

$\therefore\ \delta' = \dfrac{{\cdot}03129}{4{\cdot}225} = {\cdot}0074...$ and $y = 5{\cdot}5974$ the 2nd approximation.

Again,

$5{\cdot}597$	...	y	...	$5{\cdot}598$
$1052{\cdot}0$	...	$6y^3$	...	$1052{\cdot}56$
$789{\cdot}17$	...	$-141y$	...	$-789{\cdot}318$
$262{\cdot}83$	...	$+263$	...	$263{\cdot}14$

$\therefore\ {\cdot}31 : {\cdot}17 = {\cdot}001 : {\cdot}0005$ the 3rd correction.

Hence $y = 5{\cdot}5975$ and $\therefore\ x = -5{\cdot}5975$.

11. A root of $(x^2 + 3x + 1)^{\frac{1}{2}} + (2x^2 + 1)^{\frac{1}{5}} = 2{\cdot}617$, lies between 0 and 1.

By Double Position

0 ...	x	... 1
1 ...	$(x^2 + 3x + 1)^{\frac{1}{2}}$	... $1{\cdot}7099$
1 ...	$(2x^2 + 1)^{\frac{1}{5}}$	... $1{\cdot}2457$
2 ...	$2{\cdot}617$	... $2{\cdot}9556$

Then ${\cdot}9556 : {\cdot}617 = 1 : \delta$ a 1st correction;

$\therefore\ \delta = \dfrac{{\cdot}617}{{\cdot}9556} = {\cdot}64...$ and $x = {\cdot}64$ is a first approximation.

Again,

${\cdot}63$ ...	x	... ${\cdot}64$
$1{\cdot}48683$ ...	$(x^2 + 3x + 1)^{\frac{1}{2}}$	... $1{\cdot}49324$
$1{\cdot}12397$ ...	$(2x^2 + 1)^{\frac{1}{5}}$	... $1{\cdot}12713$
$2{\cdot}6108$ ...	$2{\cdot}617$	... $2{\cdot}62037$

Then ${\cdot}00957 : {\cdot}0062 = {\cdot}01 : \delta'$ a 2nd correction;

$\therefore\ \delta' = \dfrac{{\cdot}0062}{{\cdot}957} = {\cdot}00647$, and $\therefore\ x = {\cdot}63647$ nearly.

14. One root of the equation is found to lie between 3 and 4.

1	4	−2	10	−2	962(3·35484874
	3	21	57	201	597
	7	19	67	199	365
	3	30	147	642	299·14833
	10	49	214	841	65·85167
	3	39	264	156·1611	59·87805
	13	88	478	997·1611	5·97362
	3	48	42·537	169·4514	4·92583
	16	136	520·537	1166·6125	1·04779
	3	5·79	44·301	30·9484	·98760
	19·3	141·79	564·838	1197·5609	6019
	3	5·88	46·092	31·353	4940
	19·6	147·67	610·930	1228·914	1079
	3	5·97	8·037	2·544	988
	19·9	153·64	618·96|7	1231·45|8	91
	3	6·06	8·09	2·54	86
	20·2	159·70	627·0|6	1234·00	5
	3	1·03	8·2	·50	5
	2|0|·5	160·7|3	635·3	1234·5|0	
		1	·6	·5	
		161·7	6|3|5|·9	1|2|3|5|·0	
		1			
		1|6|3			

$\therefore$ a root is 3·35484874.

17. Now 3 is the first figure of the greatest root.

$$\text{Let } x = 3 + \frac{1}{x'},$$

and we have the transformed equation

$$5x'^3 - 7x'^2 - 6x - 1 = 0,$$

the first figure in the root of which is 2.

$$\text{Put therefore } x' = 2 + \frac{1}{x''},$$

and we have for a new transformed equation

$$x''^3 - 26x''^2 - 23x'' - 5 = 0,$$

the first figure in the root of which is 26.

Again let $x'' = 26 + \frac{1}{x'''}$,

and effect a third transformation, which will be

$$603x'''^3 - 653x'''^2 - 52x''' - 1 = 0,$$

the first figure in the root of which is 1;

$$\therefore \; x = 3 + \frac{1}{2} + \frac{1}{26} + \frac{1}{1} + \text{\&c.},$$

which furnishes the converging fractions

$$3, \frac{7}{2}, \frac{185}{53}, \frac{192}{55}, \text{\&c.}$$

18. The limiting equation is the first derived of the given equation, and is

$$\therefore \; (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b) - m^2 - n^2 - p^2 = 0,$$

$$\text{i.e. } 3x^2 - 2(a+b+c)x + ab + ac + bc = m^2 + n^2 + p^2.$$

Now, since $(a+b+c)^2$ is $> 3(ab+ac+bc)$ the roots of the limiting equation are real.

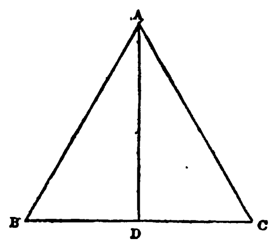
Hence the 3 roots of the proposed are all real.

GEOMETRY.

MISCELLANEOUS THEOREMS AND PROBLEMS.

1. Let ABC be the triangle; and AD the line bisecting the vertical angle BAC and meeting the base in D.

Then, in the triangles BAD, CAD, we have $BA = CA$, AD common to both the triangles, and the angle $BAD =$ angle CAD.



Therefore (Euc. I. 4), $BD = DC$,

and $\angle BDA = \angle CDA$.

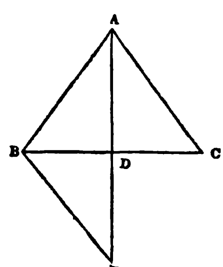
Therefore (Def. x.) AD is perpendicular to BC, and it has also been shewn to bisect BC.

2. Referring to the figure in the preceding example, since in the triangles BAD, CAD, we have $BA = CA$, $BD = DC$, and AD common to both triangles. Therefore (Euc. I. 8), $\angle BAD = \angle CAD$, or AD bisects the vertical angle; and

$\angle BDA = \angle CDA$, or AD is perpendicular to BC.

4. Let BAC be the triangle, AD the line bisecting the angle BAC, and the base BC. Produce AD to E, and make $DE = AD$, and join BE.

Now, in the triangles ADC, BDE, we have $AD = DE$ (by construction); $BD = BC$ (by hypothesis); and



$\angle BDE = \angle ADC$ (Euc. I. 15).

Therefore (Euc. I. 4) $BE = AC$,

and angle $BED =$ angle DAC;

but angle $DAC =$ angle BAD (by hypothesis);

therefore $\angle BED = \angle BAD$;

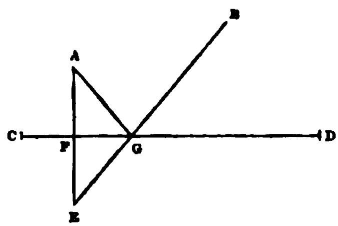
and $\therefore BE = BA$ (Euc. I. 5);

but $BE = AC$.

Therefore $BA = AC$,

or the triangle BAC is isosceles.

6. Let A, B be the two given points, and CD the given line. Draw AF perpendicular to CD, and produce it to E, making FE equal to AF, and join BE cutting CD in G. Join also AG. Then AG and BG make equal angles with CD.



For since AF is equal to FE, and FG is common to the two triangles AGF, EGF, and the included angles AFG, EFG are equal;

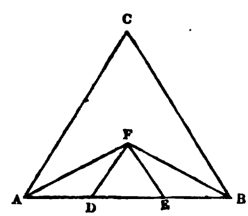
therefore the base AG is equal to the base EG, and the angle AGF to the angle EGF;

but the angle EGF is equal to the vertical angle BGD;

therefore the angle AGF is equal to the angle BGD; that is, the straight lines AG, BG make equal angles with the straight line CD.

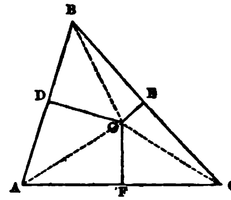
9. Let AB be the given straight line.

Upon AD describe an equilateral triangle ABC, bisect the angles at A and B by the straight lines AF, BF, meeting in F; through F draw FD parallel to AC, and FE parallel to BC. Then AB is trisected in the points D, E. For since AC is parallel to FD, and FA meets them, therefore the alternate angles FAC, AFD are equal; but the angle FAD is equal to the angle FAC, hence the angle DAF is equal to the angle AFD; and therefore DA is equal to DF. But the angle FDE is equal to the angle CAB, and FED to CBA (I. 29), and therefore the remaining angle DFE is equal to the remaining angle ACB. Hence the three sides of the triangle DFE are equal to one another, and DF has been shewn to be equal to DA;



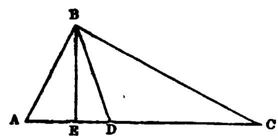
therefore AD, DE, EB are equal to one another.

13. Let D, E, F be the middle points of the sides AB, BC, AC of the triangle ACB. Draw the perpendiculars EG, FG, meeting in G. The perpendicular at D also passes through G.



Join GD, GA, GB, GC. Since $AF=FC$ and FG is common to the triangles AFG, CFG, and the angles at F are right angles, therefore AG is equal to GC. Similarly it may be shewn that GC is equal to GB, and therefore AG is equal to GB; but AD is equal to DB, and DG is common to the triangles ADG, BDG; therefore the angles at D are equal, and therefore right angles; or the perpendicular at D passes through G.

17. From B the right angle of the triangle ABC let BE be drawn perpendicular to the hypothenuse, and BD bisecting the angle ABC: the angle EBD is half the difference of the angles BAC, BCA.



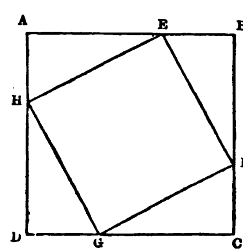
Now $\angle EBD = ABD - ABE$

$= ABD - C.$

Also $\angle EBD = EBC - DBC = A - DBC$;

$$\therefore\ EBD = \frac{1}{2}(A - C), \text{ since } ABD = DBC.$$

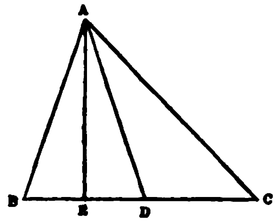
25. Let E, F, G, H be four points at equal distances from the angles of the square $ABCD$. Join EF, FG, GH, HE; $EFGH$ is also a square.



Since $AH = EB$, and $AE = BF$, and the angles at A and B right angles; therefore $HE = EF$, and angle $AEH =$ angle BFE. Similarly it may be shewn that HG and GF are each of them equal to HE and EF; therefore the figure $HEFG$ is equilateral. It is also rectangular; for since the exterior angle FEA is equal to the interior angles EBF, EFB, parts of which AEH and EFB are equal; therefore the remaining angle $FEH =$ the remaining angle FBE; and therefore is a right angle. In the same way it may be shewn that the angles at F, G, H are right angles;

and therefore $EFGH$ being equilateral and rectangular is a square.

29. From the angle A of the triangle ABC let AD be drawn to the middle point of the opposite side; the squares of AB, BC are together double the squares of AD, DB.



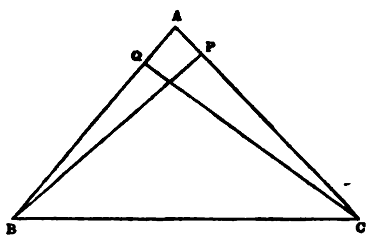
From A draw AE perpendicular to BC. Then (Eucl. II. 12)

$$AB^2 = AD^2 + DB^2 + 2BD \times DE,$$

and (Eucl. II. 13)

$$AC^2 = AD^2 + DC^2 - 2CD \times DE$$
$$= AD^2 + DB^2 - 2BD \times DE,$$
whence $AB^2 + AC^2 = 2AD^2 + 2BD^2$.

32. Let ABC be any triangle; and let BP, CQ be the perpendiculars on AC, AB.



Then (Eucl. II. 7)

$$AQ^2 - BQ^2 = AB^2 - 2AB \times BQ,$$

and (Eucl. I. 47, and Axiom 1)

$$AC^2 - BC^2 = AQ^2 - BQ^2.$$

Therefore

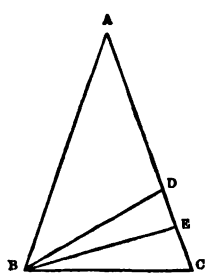
$$AC^2 = BC^2 + AB^2 - 2AB.BQ.$$

Similarly

$$AB^2 = BC^2 + AC^2 - 2AC.CP.$$

Therefore $BC^2 = AB.BQ + AC.CP$.

35. Let ABC be the triangle having angle ABC = angle ACB = twice the angle BAC. Draw BD bisecting the angle ABC, and BE perpendicular to AC; then (Eucl. I. 6)



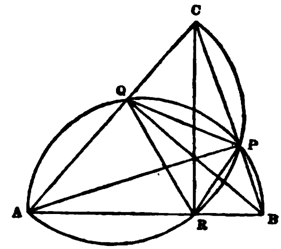
$$AD = BD = BC;$$

and therefore $DE = EC$.

Now (Eucl. II. 12)

$$AB^2 = BD^2 + AD^2 + 2AD.DE$$
$$= BC^2 + AD.(AD + 2DE)$$
$$= BC^2 + AB.BC.$$

36. Let ABC be a triangle, and AP, BQ, CR perpendiculars from the angles A, B, C on the opposite sides. The circle described on AC as diameter will pass through the points R, P, because the angles ARC, APC are right angles; hence angle APR = angle ACR = a right angle − angle CAB; and BPR = a right angle − APR = CAB. Similarly the circle described on AB as diameter will pass through the points Q, P; and $\angle APQ = \angle ABQ$ = a right $\angle$ − angle CAB.



Therefore $\angle CQP$ = a right $\angle - APQ = CAB$,

$$\text{or } \angle BPR = \angle CPQ,$$

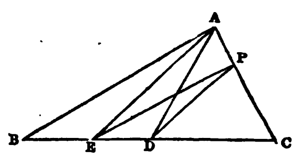
and therefore the complements of these angles are equal,

$$\text{or } \angle APR = \angle APQ.$$

Similarly it may be shewn that

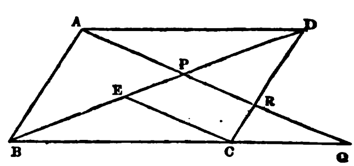
$$\angle BQP = \angle BQR, \text{ and } \angle CRP = \angle CRQ.$$

37. Let ABC be the given triangle, and P the given point. Bisect BC in D; join AD, PD; and from A draw AE parallel to PD; join PE; PE bisects the triangle ABC.



Since AE is parallel to PD, the triangle APD is equal to the triangle EPD; to each of these equals add the triangle PDC; $\therefore$ $CPE = CAD$ = half the triangle ABC.

40. Draw CE parallel to AP; and therefore equal to it; therefore also $BE = PD$, and $BP = DE$. Since CE is parallel to AR,

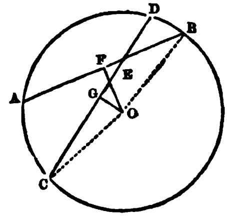


$$PR : CE \; (:: DP : DE :: BE : BP) :: CE : PQ,$$

$$\text{or } PR : AP :: AP : PQ;$$

$$\text{whence } AP^2 = PQ \, . \, PR.$$

42. Let AB, CD be the chords intersecting in E; from centre O, draw OF, OG perpendicular to AB, CD, and therefore bisecting them: then

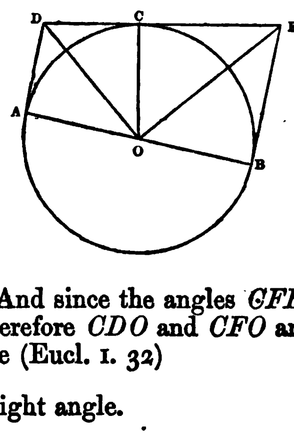


$$\text{difference of } AE, EB = 2EF,$$

$$\text{and difference of } CE, ED = 2EG: \text{ but}$$

$$\begin{aligned} AB^2 \sim CD^2 &= 4BF^2 - 4CG^2 \\ &= 4OG^2 \sim 4OF^2 \\ &= 4EF^2 \sim 4EG^2 \\ &= (2EF)^2 \sim (2EG)^2 \\ &= (AE - EB)^2 \sim (CE - ED)^2. \end{aligned}$$

46. Let AB be any diameter of the circle, and AD, BF tangents at A, B; also DF a tangent at any point C: and let O be the centre; join DO, FO; the angle DOF is a right angle.

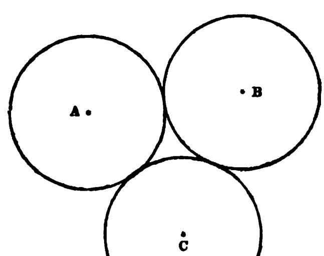


Join CO. Then since $CF = FB$, $CO = OB$; and the angles at C and B being right angles, are equal, therefore $\angle CFO = \angle OFB$, and CFB is bisected by FO.

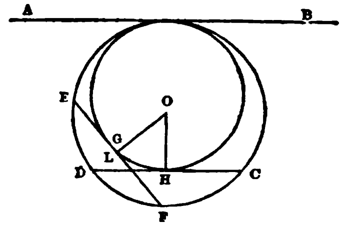
Similarly it may be shewn that the angle ADC is bisected by DO. And since the angles CFB, CDA are equal to two right angles, therefore CDO and CFO are equal to one right angle, and therefore (Eucl. I. 32)

the angle DOF is a right angle.

48. Let A be the centre of the central circle, and B, C the centres of two others. Then, the triangle ABC being equilateral, the angle BAC is one-third of two right angles; and (Eucl. I. 15, Cor. 2) there may be six such angles around the point A; and therefore, six circles can be placed round the given circle.



50. Let AB be the common tangent to the two circles. Draw CD parallel to AB, touching the inner circle, and let EF be any other tangent to the inner circle. From O, the centre of the outer circle, draw OH perpendicular to CD, and OL perpendicular to EF, cutting the circle in G. Then OH is less than OG (Eucl. III. 7), therefore OH is less than OL; and therefore BC is greater than EF. Similarly, if any other line were taken, it might be shown that BC is greater than that line; therefore, &c.

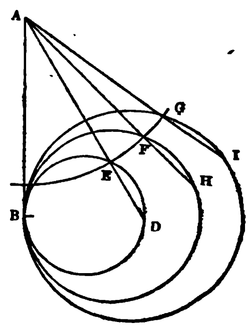


53. Let the circles touch each other in the point B, to which let a tangent BA be drawn, and from any point A in it as centre, with any radius, let a circle EFG be described. Draw the lines AED, AFH, AGI; then will the parts DE, HF, IG be equal.

For since AB touches the circle (Eucl. III. 36),

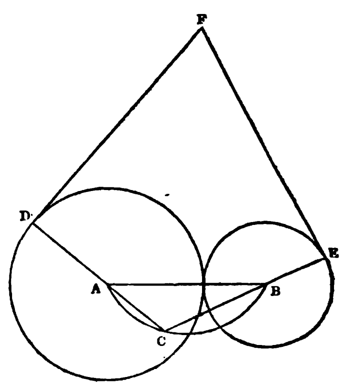
$$DA : AB :: AB : AE,$$

for the same reason $AB : AH :: AF : AB$; therefore, ex æquo, $DA : AH :: AF : AE$: but $AF = AE$, therefore $DA = AH$; and therefore $DE = HF$. In the same way it may be proved that $IG = HF = DE$.



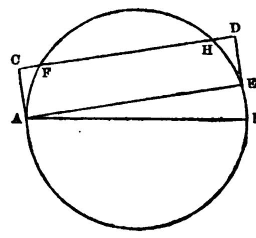
55. Let A, B be the centres of the given circles; join AB; then AB will pass through the point of contact. On AB describe a segment containing an angle equal to the supplement of the given angle, in which arc take any point C; join CA, CB, and produce them to meet the circumferences in D and E; at which points let the tangents DE, EF be drawn; then the angle F at the intersection of the tangents will be equal to the given angle.

For FD, FE being tangents at D and E, each of the angles ADF, BEF is a right angle; there-



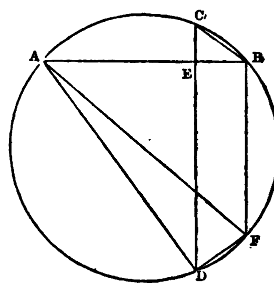
fore the angles DFE and DCE together are equal to two right angles, or DFE is the supplement of DCE, and is therefore equal to the given angle.

57. From A and B, the extremities of any diameter AB, let fall the perpendiculars AC, BD on any chord FH; and let the perpendicular BD cut the circumference of the circle in E; then $ED = AC$.



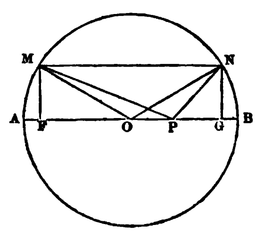
Join AE; then (Eucl. III. 31) the angle AEB is a right angle; therefore AE is parallel to CD; and AC is parallel to ED; therefore the figure $ACED$ is a parallelogram; and therefore $AC = ED$.

60. Let AB, CD cut one another at right angles in E; the sum of the squares of AE, EB, CE, ED will be equal to the square of the diameter.



Draw the diameter AF. Join FB, BC, FD, DA; then ABF being a right angle is equal to AED, and therefore BF is parallel to CD, and (Ex. 59) $BC = FD$. And since the angles at E are right angles, the squares of CE, EB are equal to the square of CB; i.e. to the square of DF; but the squares of AE, ED are equal to the square of AD; therefore the squares of CE, EB, AE, ED are equal to the squares AD, DF; i.e. to the square of AF, ADF being a right angle.

62. Let MN be drawn parallel to AB the diameter of the circle AMN; and from any point P in AB let PM, PN be drawn. Take O the centre of the circle, and join MO, NO; and let fall the perpendiculars MF, DG on AB. Then since MN is parallel to AB, the angles AOM, BON are equal, and $OF = OG$.



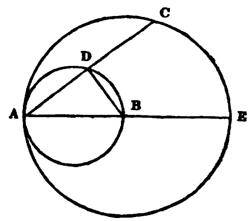
Now (Eucl. II. 12),

$$MP^2 = MO^2 + OP^2 + 2OF \times OP,$$

and (Eucl. II. 13) $NP^2 = NO^2 + OP^2 - 2OG \times OP$,

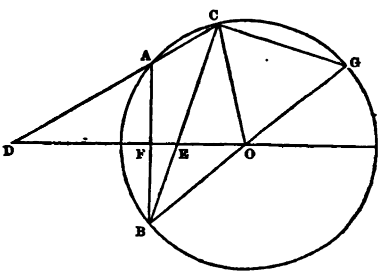
whence the squares of MP, NP are equal to twice the squares of MO, OP, or twice the squares of AO, OP; i.e. to the squares of AP, PB.

63. Let ADB be a circle described on the radius AB of the circle ACE. Draw any line AC meeting the circle ABD in D; AD is equal to DC.



Join DB. Then the angle ADB being in a semicircle is a right angle; and therefore BD being drawn from the centre B of the circle ACE bisects AC. (Eucl. III. 3.)

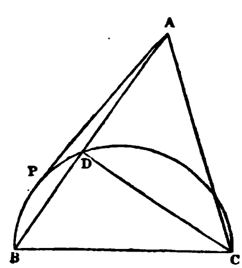
65. Let BAC be a circle; AB any chord in it; and C any point in its circumference. Let AC, BC meet a diameter perpendicular to AB in D and E. Take O the centre of the circle; draw the diameter BOG. Join GC, CO. Since the angle OCB is equal to the angle OBC, and BGC to FAD, and that CBG and BGC together are equal to a right angle; therefore OCE and FAD together are equal to a right angle, and therefore to FAD and ADF together; hence OCE is equal to ADO; therefore the triangles COD, COE are equiangular,



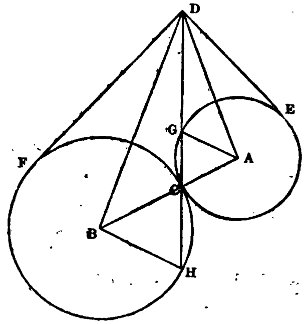
$$\text{and } DO : OC :: OC : OE;$$

$$\text{therefore } OD \,.\, OE = OC^2 = OA^2.$$

67. Let ABC be a triangle whose acute vertex is A. On BC describe the circle BDC, cutting the side AB of the triangle in D; join CD; and from A draw AP a tangent to the circle at P. Then (Eucl. III. 31) the angle CDB is a right angle; and (Eucl. II. 13) BC^2 is less than $AB^2 + AC^2$ by $2AB \,.\, AD$; i.e. (Eucl. III. 36) by $2AP^2$.



68. Let CFH, CGE be the circles touching each other in the point C, A and B their centres, and D the point without them, such that the angle ADC is equal to the angle CDB. Let DC cut the circles in G and H. Join AG, BH, and draw the tangents DE, DF. Then



$$DE^2 = DC.DG, \text{ and } DF^2 = DC.DH;$$

also the triangles DGA, DCB being similar, we have

$$BC : DC = AG : DG,$$

$$\text{or } BC : DC = AC : DG.$$

And for the same reason we have

$$AC : DC = BH : DH,$$

$$\text{or } AC : DC = BC : DH;$$

$$\text{therefore } BC.AC : DC^2 = BC.AC : DG.DH;$$

$$\text{therefore } DG : DC = DC : DH,$$

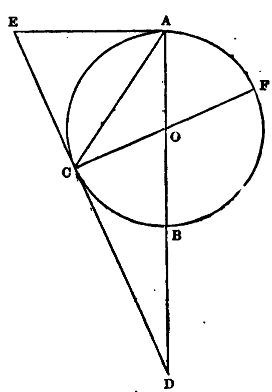
$$\text{or } DG.DC : DC^2 = DC^2 : DH.DC,$$

$$\text{or } DE^2 : DC^2 = DC^2 : DF^2,$$

$$\text{and } DE : DC = DC : DF;$$

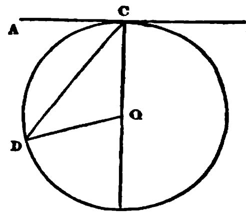
$$\text{therefore } DE.DF = DC^2.$$

70. From the point E without the circle let EA, ECD be drawn touching the circle in A and C, and let ED meet the diameter AB, drawn from A, in the point D. Join AC; the angle AEC is double of CAB.



Through C draw the diameter COF; then the angle FCD is a right angle, and therefore equal to EAD, and EDA is common to the triangles EDA, COD, therefore the angle COD is equal to AED. But COB is (Eucl. III. 20) double of CAD; therefore the angle AEC is double of CAD.

73. Let AB be the given straight line, C the given point in which the circle is to touch it, D the point through which it must pass. Draw CO perpendicular to AB. Join CD, and at the point D make the angle ODO = angle DCO; the intersection of the lines CO and DO is the centre of the circle required.

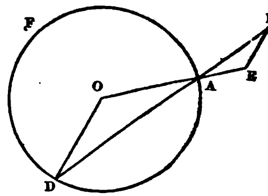


Since the angle

$$DCO = CDO, \quad CO = DO,$$

and therefore a circle described from the centre O, at the distance OD, will pass through C, and touch the line AB in C, because OC is perpendicular to AB.

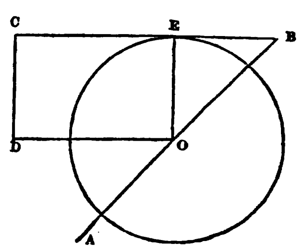
75. Let A be the given point in the circumference of the circle whose centre is O; B the given point without. Join BA and produce it to D. Join OD; and through A draw OAE; and draw BE parallel to OD, cutting OAE in E. E is the centre of the circle required.



Since (Eucl. I. 29) the angle ODA is equal to ABE, and OAD to BAE, therefore the triangles ODA, ABE are similar, and OD being equal to OA, AE will be equal to EB; therefore a circle described with the centre E, and radius EA, will pass through B, and touch the circle ADF in the point A, since the line joining the centres passes through A.

78. Let AB be the given line in which the centre is to be; BC the line which the circle is to touch.

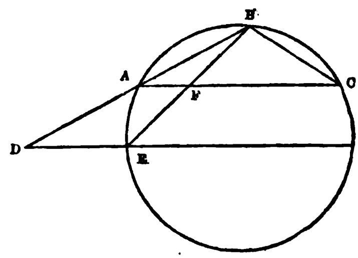
In BC take any point C, and draw CD at right angles to it; and make CD equal to the given radius. Through D draw DO parallel to CB; O is the centre of the circle required.



Through O draw OE parallel to DC; therefore CO is a parallelogram; whence $OE = DC$, i.e. to the given radius. With the centre O,

and radius OE, describe a circle; it will touch CB in E, because CO being a parallelogram, and ECD a right angle, CEO is also a right angle.

80. Let ABC be a triangle having the obtuse angle ABC. Describe a circle about it, and produce BA to D, making $AD = AB$. From D draw DE parallel to AC, meeting the circle in E; join BE, cutting AC in F; BF will be a mean proportional between AF and FC.



For (Eucl. VI. 2)

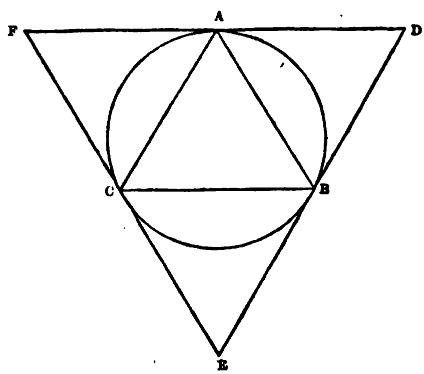
$$BF : FE :: BA : AD,$$

$$\text{and since } BA = AD; \therefore BF = FE.$$

Now (Eucl. III. 35) $AF . FC = BF . FE = BF^2$;

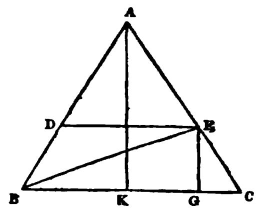
$$\therefore AF : FB = FB : FC.$$

84. Let ABC be an equilateral triangle inscribed in a circle, about which another, DEF, is circumscribed, touching the circle in the points A, B, C.



Since DA touches the circle, the angle $DAB = ACB$ (Eucl. III. 32); but $ACB = ABC$; $\therefore DAB = ABC$, and they are alternate angles, therefore DF is parallel to BC. In the same manner it may be shewn that AB is parallel to FE; therefore $ABCF$ is a parallelogram; and the triangle ABC is equal to AFC. In the same manner ABC may be shown to be equal to each of the triangles ABD, BCE; and therefore it is one-fourth of the circumscribing triangle.

87. Let ABC be an isosceles triangle. Draw DE parallel to BC; join BE. Then



$$BE^2 = BC.DE + CE^2.$$

Draw EG perpendicular to BC.

Then (Eucl. II. 13)

$$BE^2 = CE^2 - BC^2 + 2BC.BG$$
$$= CE^2 + BC(2BG - BC).$$

Draw AK perpendicular to BC;

then (Eucl. VI. 2) $KG : GC = AE : EC$,

or $BC - 2GC : 2GC = AE : EC$;

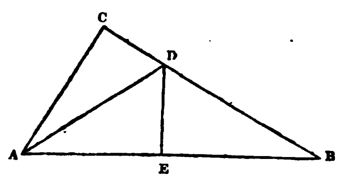
$\therefore BC - 2GC : BC = AE : AC = DE : BC$;

$\therefore BC - 2GC = DE$,

or $BC - 2(BC - BG)$, i.e. $2BG - BC = DE$;

$\therefore BE^2 = CE^2 + BC.DE$.

89. Let ABC be a triangle having a right angle C. Draw AD bisecting the angle A and meeting BC in D; then



$$DE = DC.$$
$$AC^2 : AD^2 = BC : 2BD.$$

Draw DE perpendicular to AB.

Then (Eucl. VI. 3) $AC : CD = AB : BD$,

and (Eucl. VI. 4) $AB : BC = BD : BE$,

or $AB : BD = BC : BE$;

therefore $AC : CD = BC : BE$,

and $AC^2 : CD^2 = BC^2 : BE^2$;

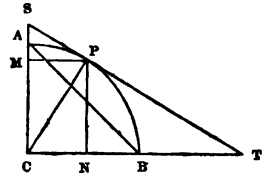
therefore $AC^2 : AD^2 = BC^2 : BE^2 + BC^2$,

also $BE^2 = BD^2 - DE^2 = BD^2 - DC^2 = (BD + DC)(BD - DC)$
$= BC(BD - DC)$;

whence it follows that

$$AC^2 : AD^2 = BC^2 : BC(BD - DC + BC) = BC : 2BD.$$

91. Let the tangent SPT meet the radii CA, CP produced in S and T. Draw PM at right angles to AC and PN to BC. Then



$$\text{triangle } SCT : \text{triangle } ACB = CS \,.\, CT : CA \,.\, CB,$$

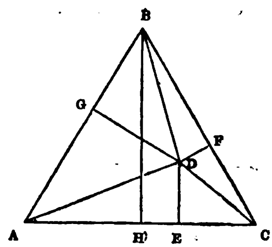
$$\text{also } CS : CP = CP : CM,$$

$$\text{and } CT : CP = CP : CN;$$

$$\text{therefore } CS \,.\, ST : CP^2 = CP^2 : CN \,.\, CM = CP^2 : CM \,.\, MP;$$

$$\text{therefore triangle } SCT : \text{triangle } ACB = CP^2 : CM \,.\, MP$$
$$= \text{triangle } ACB : \text{triangle } CMP.$$

94. From any point D within the equilateral triangle ABC, let perpendiculars DE, DF, DG be drawn to the sides; they are together equal to BH a perpendicular drawn from B on the opposite side AC.



Join DA, DB, DC. Since triangles upon the same and equal bases are to one another as their altitudes; therefore

$$ABC : ADC = BH : DE,$$

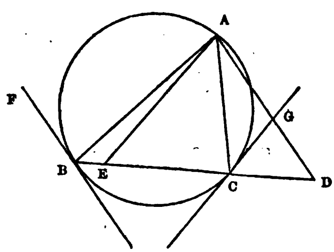
$$\text{also } ABC : BDC = BH : DF,$$

$$\text{and } ABC : ADB = BH : DG;$$

whence $ABC : ADC + BDC + ADB = BH : DE + DF + DG$, in which proportion the first term being equal to the second;

$$\text{therefore } DE + DF + DG = BH.$$

95. Let ABC be the inscribed triangle. Draw BF, CG tangents at B and C; and from A draw AD, AE parallel to BF, CG respectively, meeting the base BC in D and E.



The angle

$$ABF = \angle BAD = \angle ECA$$

also the angle

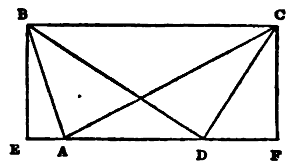
$$ACG = \angle ABD = \angle CAE;$$

therefore the triangles BAD, ACE are similar, and

$$\angle ACE = \angle ADB; \quad \therefore AD = AE.$$

Hence $BD : CE = \Delta BAD : \Delta ECA = AB^2 : AC^2$, AB, AC being homologous sides of similar triangles.

98. Suppose A and D to fall both *within* the perpendiculars BE, CF; then



$$AC^2 = AD^2 + CD^2 + 2AD \cdot FD;$$

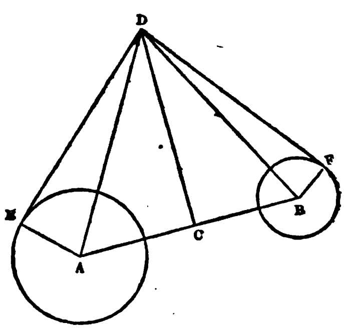
and $BD^2 = AD^2 + AB^2 + 2AD \cdot AE$;

therefore

$$AC^2 - BD^2 : CD^2 - AB^2 = CD^2 - AB^2 + 2AD(DF - AE) : CD^2 - AB^2,$$

or $$AC^2 - BD^2 : CD^2 - AB^2 = DF^2 - AE^2 + 2AD(DF - AE) : DF^2 - AE^2$$
$$= DF + AE + 2AD : DF + AE$$
$$= BC + AD : BC - AD.$$

101. Let D be any point in the perpendicular to AB, through C; draw the tangents DE, DF to the circles, and join DA, DB.



Since $AC + CB : R + r = R - r : AC - CB$;

therefore $AC^2 - CB^2 = R^2 - r^2$,

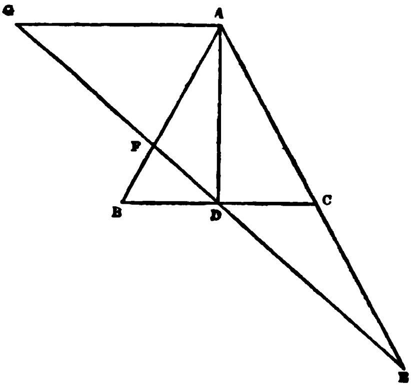
and $AC^2 - R^2 = BC^2 - r^2$.

Therefore $$DE^2 = AD^2 - AE^2 = CD^2 + AC^2 - R^2$$
$$= CD^2 + BC^2 - r^2 = BD^2 - r^2 = DF^2.$$

Therefore $DE = DF$.

104. Let the angle BAC be bisected by the line AD; and through any point D in this line draw $GFDE$ meeting AG a perpendicular to AD in G, and the sides in F and E; then will

$$GE : GF = ED : FD.$$



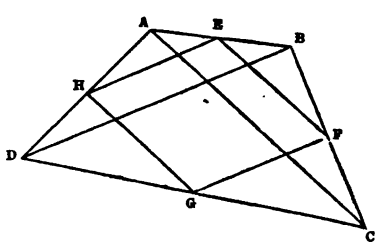
For through D draw BC parallel to AG, and therefore perpendicular to AD; then the angles BDA, CDA being right angles are equal, and $BAD = CAD$, and AD is common to the triangles BDA, CDA; therefore $BD = DC$. But DC being parallel to AG,

$$GE : ED :: GA : DC :: GA : BD :: GF : FD,$$

since the triangles GFA, BFD are similar;

therefore $GE : GF = ED : FD$.

105. Let $ABCD$ be a quadrilateral, whose sides are bisected in E, F, G, H. Let the points of bisection be joined, and draw the diagonals AC, BD.

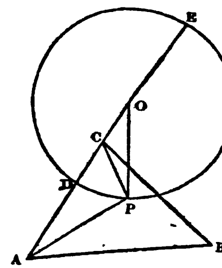


Since AB, AD are bisected in E and H (Eucl. VI. 2), EH is parallel to BD; and for the same reason FG is parallel to BD, and therefore to EH. In the same way it may be shown that EE is parallel to HG, and therefore the figure $EFGH$ is a parallelogram.

Again (Eucl. VI. 19), the triangle EBF is to the triangle ABC in the duplicate ratio of EB to AB, i.e. in the ratio of

1 : 4, therefore EBF is equal to one-fourth of ABC; for the same reason HDG is one-fourth of DAC, whence EBF and HDG are together equal to one-fourth of the quadrilateral. For the same reason HAE and GFC together are equal to one-fourth of the quadrilateral; therefore the four triangles together are equal to half the quadrilateral; and consequently $HEFG$ is equal to half of $ABCD$.

107. Let ACB be an isosceles triangle, having the angle A equal to the angle B.



Take the point D in AC, such that $AD = 2DC$. Produce AC to E until $AE = 2AC$, and on DE as diameter describe a circle, in which take any point P; then $AP = 2CP$.

Join OP, CP, AP. From the construction it is evident

$$AO = 2DO, \text{ and } DO = 2CO,$$

and consequently $AO \times OC = DO^2 = OP^2$,

$$\text{or } AO : OP = OP : OC;$$

therefore the triangles AOP, POC are equiangular;

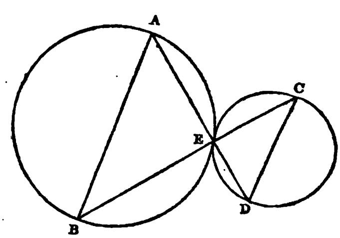
$$\text{therefore } AP : PO = PC : CO,$$

$$\text{or } AP : PC = PO : CO,$$

$$\text{or } AP : PC = DO : CO = 2 : 1;$$

$$\text{therefore } AP = 2PC.$$

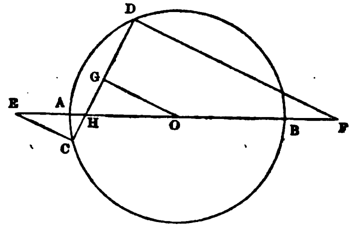
109. Let AEB, CED be two circles touching each other in E, and a straight line AC in A and C; draw the diameters AB, CD; AC is a mean proportional between AB and CD.



Join AD, BC; these lines (Ex. 108) pass through the point of contact E. And since CA touches the circle in A, from which point AE is drawn, the angle CAD is equal to the angle ABE in the alternate segment; also the angle ACD being a right angle is equal to the angle CAB; therefore the triangles ACD, ABC are equiangular, and

$$BA : AC = AC : CD.$$

113. At C and D the extremities of the chord CD, let perpendiculars to it be drawn meeting a diameter AB in E and F; E and F are equally distant from the centre O.



Draw OG perpendicular to CD, and therefore bisecting it, then OG is parallel to DF; whence

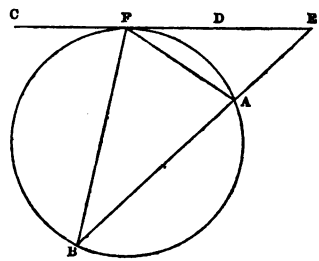
$$GD : OF = HG : HO = HC : HE,$$

since the triangles HGO, HEC are equiangular;

therefore (Eucl. v. 18, 15) $DG : OF = GC : OE$,

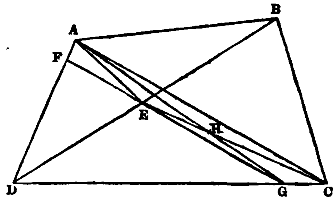
but $GD = GC$; therefore $OF = OE$.

115. Let A, B be the given points, and CD the given straight line. Join BA, and produce it to meet CD in E. Take EF a mean proportional between EA and EB. Join FA, FB, and describe a circle about the triangle AFB; it will be the circle required.



Since EF is a mean proportional between EA and EB, EF touches the circle (Eucl. III. 37) which passes through A and B.

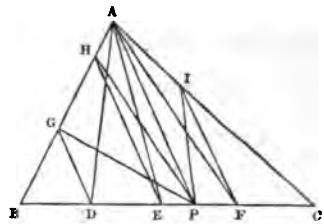
117. Let $ABCD$ be the given trapezium, and A the angle from which the bisecting line is to be drawn. Draw the diagonals AC, BD; and bisect BD which is opposite to the angle A in the point E. Join AE, CE; and through E draw FEG parallel to AC. Join AG; AG bisects the trapezium.



Since $DE = EB$, the triangles AED, AEB are equal; as also DEC, BEC; therefore the figure $AECD$ is equal to the figure $AECB$. Also (Eucl. I. 38) the triangles AEG, CEG are equal; take away therefore the com-

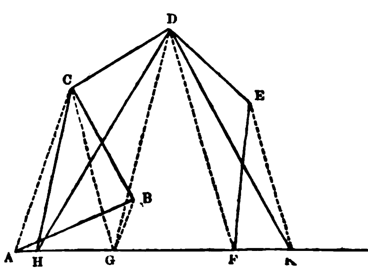
mon part EHG, and $AEH = GHC$. To the figure $AECD$ add AEH, and take away its equal GHC; and to $AECB$ add GHC, and take away its equal AEH; and the triangle AGD is equal to the trapezium $AGCB$, or the given trapezium is bisected by AG.

118. Let ABC be the given triangle, and P the given point in the side BC. Divide BC in the points D, E, F, &c. into the required number of equal parts. Join AD, AE, AF, AP; and draw DG, EH, FI parallel to AP. Join PG, PH, PI; they will divide the triangle into the required number of equal parts.



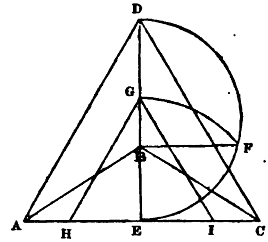
For the triangles ABD, ADE, AEF, AFC, being as their bases will be equal. And since DG is parallel to AP, the triangles DGA, DGP are equal, therefore $DBA = GPB$. And since the triangle $ADP = AGP$, and $AEP = AHP$, therefore $ADE = HPG$. Also $APE = AHP$, and $APF = AIP$, therefore $AEF = AHPI$, and $AFC = PIC$; therefore the parts PBG, GPH, $HPIA$, IPC are equal to ABD, ADE, AEF, AFC, and are consequently equal to each other. The same may be proved whatever be the number of parts.

119. Let $ABCDE$ be the given rectilineal figure, and D the angle which is to be the vertex of the triangle, the base being in the side AF. Join CA, and draw BG parallel to it; join CG, DG, DF. Draw CH parallel to DG. Join DH. Draw EK parallel to DF, meeting AF in K. Join DK; HDK will be equal to $ABCDEF$.



Since BG is parallel to CA, the triangles BAG, BCG are equal; the figure therefore is equal to $GCDEF$. And since GD is parallel to CH, the triangles GCD, GHD are equal. Again, since EK is parallel to DF, the triangles DEF, DKF are equal; whence the whole figure $ABCDEF$ is equal to the triangles DHG, DGF, DKF, that is, to the triangle DHK.

121. Let ABC be the given isosceles triangle. On AC describe an equilateral triangle ADC, and from D draw DE perpendicular to AC; it will also bisect AC and pass through B. On DE describe a semicircle; and from B draw BF perpendicular to DE, meeting the circle in F; with centre E, and radius EF, describe a circle meeting ED in G; draw GH, GI parallel to DA, DC respectively; the triangle GHI is equilateral, and equal to ABC. Since GH is parallel to AD, and GI to DC, the triangles GHI, ADC are similar; but ADC is equilateral; and therefore also GHI is equilateral.



$$\text{Also (Eucl. VI. 8. Cor.) } ED : EG = EG : EB,$$

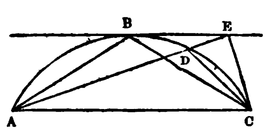
$$\text{and (Eucl. VI. 2) } ED : EG = EA : EH;$$

$$\therefore EG : EB = EA : EH;$$

and therefore (Eucl. VI. 15) the triangles EGH, EBA are equal. But $GHE = GIE$, and $BAE = BCE$;

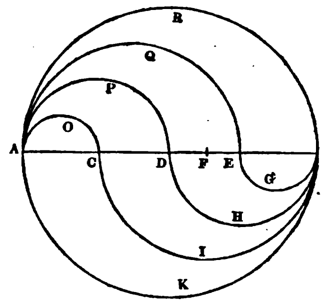
$$\text{therefore also } GHI = BAC.$$

124. Let AC be the given base. On it describe a segment containing an angle equal to the given angle; bisect the arc ABC in B, and join AB, BC. The triangle ABC is a maximum.



Through B draw BE parallel to AC, and therefore a tangent to the circle at B. Take any point D in the circumference of the circle; join AD, and produce it to meet BE in E; join CD, CE. Now the triangle AEC is equal to the triangle ABC on the same base and between the same parallels; and the triangle ADC has the same vertical angle as the triangle ABC; but AEC is greater than ADC; therefore the isosceles triangle ABC is greater than ADC.

126. Let it be required to divide the circle $AKBR$ into any number, (suppose *four*) of equal parts. Divide the diameter AB into the required number of equal parts in the points C, D, E. On AC, AD, AE describe the semicircles AOC, APD, AQE; and on BC, BD, BE, on the opposite side of the diameter, describe the semicircles BIC, BHD, BGE. The circle $AKBR$ will be divided into four equal parts by the equally compounded semi-circumferences $AOCIB, APDHB, AQEGB$.



For $AB : AD =$ semi-circumference ARB : semi-circ. APD,

and $AB : BD =$ semi-circumference ARB : semi-circ. BHD.

$$\text{Hence } AB : AD + BD = ARB : APD + BHD,$$

$$\text{but } AD + BD = AB; \therefore\ APD + BHD = ARB,$$

i.e. the compounded boundary $APDHB$ is equal to the semi-circumference ARB of the original circle.

In the same manner it may be shewn that each of the compounded boundaries $BGEQA$, $BICOA$ is equal to the semi-circumference ARB; and the same is true whatever be the number of divisions.

$$\text{Again, the circle on } AB : \text{circle on } AD = AB^2 : AD^2;$$

$$\text{also, the circle on } AB : \text{circle on } AE = AB^2 : AE^2;$$

$$\text{hence, circle on } AD : \text{circle on } AE = AD^2 : AE^2;$$

therefore, circle on $AE -$ circle on AD : circle on AE

$$= AE^2 - AD^2 : AE^2;$$

therefore,

circle on AB : 2 space $APDEQA = AB^2 : AE^2 - AD^2$

$$= AB^2 : (AE + AD)(AE - AD).$$

Bisect the line DE in F;

then $AE + AD = 2AF$, and $AE - AD = DE$; therefore

$$\text{circle on } AB : 2 \text{ space } APDEQA = AB^2 : 2AF \times DE,$$

or, circle on AB : space $APDEQA = AB^2 : AF \times DE$;

also, circle on AB : space $BGEDHB = AB^2 : BF \times DE$;

hence, circle on AB : space $APDHBGEQ$

$$= AB^2 : AF \times DE + BF \times DE$$

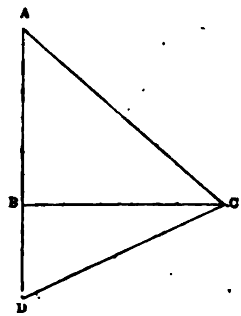
$$= AB^2 : AB \times DE$$

$$= AB : DE;$$

but DE is equal to one-fourth AB; therefore the space $APDHBGEQ$ is one-fourth of the whole circle on AB.

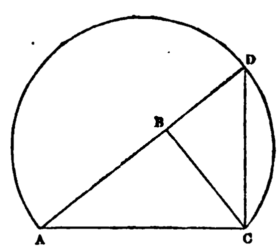
Similarly, it may be shown that each of the other spaces is equal to one-fourth of the circle on AB. The same is evidently true whatever be the number of equal parts.

129. Let BC be the given base. Draw BA at right angles to BC; produce AB to D, and make BD equal to the difference between the hypothenuse and the other side. Join DC. Make the angle DCA equal to the angle BDC. ABC is the triangle required.



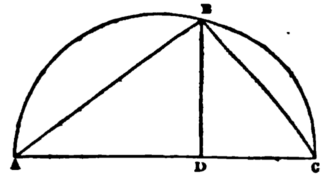
The angle ADC being equal to the angle ACD, the side AC is equal to AD, and the difference between AC and AB is equal to BD; i.e. to the given difference.

131. Let AC be the given hypothenuse. On AC describe a segment of a circle, containing an angle equal to half a right angle, in which place the straight line AD equal to the sum of the sides. Join DC, and make the angle DCB equal to the angle ADC. Then ABC is the triangle required.



The angle ABC is equal to the two interior angles BDC, DCB, i.e. to twice the angle ADC, and is therefore a right angle; also $AB + BC$ is equal to AD, i.e. to the sum of the sides.

134. Let AC be the given hypothenuse; divide it in the point D in extreme and mean ratio. On AC describe the semicircle ABC, and from D draw DB at right angles to AC to meet it. Join AB, BC, ABC is the triangle required. By construction,



$$\text{we have } AC : AD = AD : DC,$$

$$\text{and (Eucl. VI. 8. Cor.) } AC : BC = BC : DC;$$

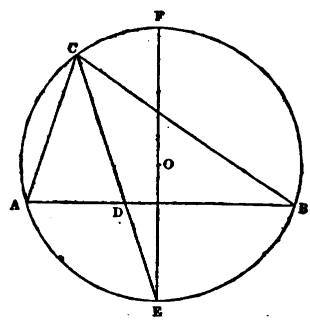
$$\text{therefore } BC = AD.$$

$$\text{Again, since } AC : AB = AB : AD;$$

$$\text{therefore } AC : AB = AB : BC;$$

therefore the sides of the triangle are in continued proportion.

139. Let AB be the given base; on it describe a segment ACB containing an angle equal to the given vertical angle, and complete the circle. Draw the diameter FE perpendicular to AB, and therefore bisecting the arc AEB in the point E. Divide AB in the point D in the ratio of the sides; join ED, and produce it to meet the circle in C. Join AC, BC; ABC is the triangle required.



By construction the angle ACB is equal to the given vertical angle; and (Eucl. VI. 33) the angle ACD is equal to the angle DCB, or the angle ACB is bisected by CD; therefore (Eucl. VI. 3) $AC : CB :: AD : DB$, i.e. in the given ratio.

MENSURATION.

Areas of Plane Figures.

Ex. 1.

6. The height $= 4 \times \sin 16^\circ 43 = 4 \times \cdot 287639 = 1 \cdot 150556$ ch.

$\therefore$ area $= 1 \cdot 150556 \times 5 \cdot 5 = 6 \cdot 32801$ sq. ch. $=$ 2 r. 21 p. $7\frac{1}{2}$ yd.

8. $$2\tfrac{1}{2}d. = £\tfrac{1}{96}; \text{ and } £33\ 16s.\ 10\tfrac{1}{2}d. = £33 \cdot 84375;$$

$$\therefore \text{ area of garden} = 33 \cdot 84375 \times 96 = 3249 \text{ sq. yd.};$$

$$\therefore \text{ side of square} = \sqrt{3249} = 57 \text{ yd.}$$

13. $$5 \text{ guineas} = 1260d.;$$

$$\therefore \text{ area of rhombus} = \frac{1260}{7} = 180 \text{ sq. yd.};$$

$$\therefore \text{ height} = \frac{180 \times 9}{45} = 36 \text{ feet};$$

$$\therefore \text{ required angle} = \sin^{-1}\left(\frac{36}{45}\right) = \sin^{-1}\left(\frac{4}{5}\right) = 53^\circ\, 7'\, 48''.$$

Ex. 2.

8. The remaining angle of the triangle is $83^\circ 30'$.

Let $x =$ the side opposite to this angle; then

$$x : 72 = \sin 83^\circ 30' : \sin 38^\circ 40';$$

$$\therefore x = 72 \cdot \frac{\sin 83^\circ 30'}{\sin 38^\circ 40'};$$

$$\therefore \text{ area} = \frac{1}{2} \times 72 \times 72 \cdot \frac{\sin 83^\circ 30'}{\sin 38^\circ 40'} \sin 57^\circ 50' = 3489 \cdot 224.$$

11. $$\text{Area of field} = \frac{7{\cdot}38 \times 5{\cdot}83}{2 \times 10} = 2{\cdot}15127 \text{ acres};$$

$$\therefore \text{rate per acre} = \frac{12}{2{\cdot}15127} = £5{\cdot}5781$$
$$= £5\ 11s.\ 6\tfrac{3}{4}d.$$

13. If x be the required angle, we have

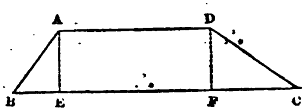
$$\frac{1}{2} \times (275)^2 . \sin x = 6 \text{ acres} = 6 \times 4840 \text{yd.};$$

$$\therefore \sin x = \frac{58080}{(275)^2} = {\cdot}768 = \sin 50^\circ\ 10'\ 28'';$$

$$\therefore x = 50^\circ\ 10'\ 28''.$$

Ex. 3.

2. Let $ABCD$ be the trapezoid. Draw $AEDF$ perpendicular to BC. Let $BE = x$; then



$$FC = 33\tfrac{1}{3} - x;$$

$$\therefore 20^2 - x^2 = (26\tfrac{2}{3})^2 - (33\tfrac{1}{3} - x)^2;$$

$$\therefore 400 = \frac{6400}{9} - \frac{10000}{9} + \frac{200x}{3};$$

$$\therefore \frac{200}{3} x = \frac{7200}{9} \text{ or } x = 12 \text{ ft.};$$

$$\therefore AE = d = (20^2 - x^2)^{\frac{1}{2}} = 16 \text{ ft.};$$

$$\therefore \text{area} = 8 \times (72 + 38\tfrac{2}{3}) = 8 \times 110\tfrac{2}{3} = 885\tfrac{1}{3} \text{ ft.}$$

3. Referring to the figure in the last example, we have

$$BE = AE \cot 54^\circ\ 12' = 36 \times {\cdot}7212227 = 25{\cdot}964.$$

Similarly $FC = AE \cot 46^\circ\ 15' = 36 \times {\cdot}957292 = 34{\cdot}4625;$

$$\therefore AD = BC - (BE + FC) = 137 - 60{\cdot}4265 = 76{\cdot}5735;$$

$$\therefore \text{area} = \frac{1}{2} \times 36 \times (137 + 76{\cdot}5735) = 18 \times 213{\cdot}5735 = 3844{\cdot}323.$$

9. As $122° \ 30'$ is the supplement of $57° \ 30'$, we have

$$\text{area} = \frac{1}{2}\{690 \times 467 + 428 \times 359\} \sin 57° \ 30'$$

$$= 237941 \times \cdot 84339 = 200677 \cdot 06 \text{ yds.}$$

$$= 41 \cdot 4622A.$$

11. $$\therefore DE = \sqrt{220^2 - 100^2} = \sqrt{38400} = 195 \cdot 96,$$

$$\text{and } BF = \sqrt{265^2 - 70^2} = \sqrt{65325} = 255 \cdot 588;$$

$$\therefore \text{area} = \frac{1}{2} \times 378 \times (195 \cdot 96 + 255 \cdot 588) = 189 \times 451 \cdot 548$$

$$= 85345 \cdot 572 \text{yd.} = 17 \cdot 6333 \text{ acres} = 17 \text{ ac. } 2 \text{ r. } 21 \text{ p.}$$

12. Let $ABCD$ be the trapezium, AC being the diagonal, then

$$AC = \sqrt{335^2 + 426^2} = \sqrt{293701} = 541 \cdot 942,$$

$$\text{area of } \Delta ABC = \frac{1}{2} \times 335 \times 426 = 71355 \text{ yd.}$$

and area of ΔADC

$$= \sqrt{624 \cdot 971 \times 83 \cdot 029 \times 237 \cdot 971 \times 303 \cdot 971} = 61266 \cdot 484;$$

$\therefore$ whole area $= 61266 \cdot 484 + 71355 = 132621 \cdot 484$ yd. $= 27 \cdot 401$ a.

Ex. 4.

9. The formulæ are,

$$r = a \cdot \cot \frac{180°}{n}, \quad R = a \cdot \operatorname{cosec} \frac{180°}{n},$$

and here $n = 5$ and $a = \frac{3}{2}$;

$$\therefore r = \frac{3}{2} \cot 36° = \frac{3}{2} \times 1 \cdot 376382 = 2 \cdot 06457,$$

$$\text{and } r = \frac{3}{2} \operatorname{cosec} 36° = \frac{3}{2} \times 1 \cdot 701302 = 2 \cdot 55195.$$

13. The angle subtended at the centre by each side of the nonagon is $\frac{360^\circ}{9} = 40^\circ$.

And since the radius of the circumscribing circle is 12, the area of each of the 9 triangles into which the polygon is divided by lines drawn from the centre of the circle to its angular points is

$$\frac{1}{2} \times 12^2 \sin 40^\circ = 72 \times \cdot 642788 = 46 \cdot 2807;$$

$$\therefore \text{ area of polygon} = 9 \times 46 \cdot 2807 = 416 \cdot 526.$$

14. Side of polygon $= 2r \tan \frac{180^\circ}{n} = 20 \times \tan \frac{180^\circ}{25}$;

$$\therefore \text{ area of each of the 25 triangles} = 10^2 \tan 7^\circ\ 12';$$

$$\therefore \text{ area of polygon} = 2500 \times \tan 7^\circ\ 12' = 315 \cdot 823.$$

Ex. 5.

4. Circumference $= 2\pi r = 1760$ yd.;

$$\therefore r = \frac{880}{\pi},$$

$$\text{area} = \pi r^2 = \frac{(880)^2}{\pi} = 246498 \cdot 623 \text{ yd.}$$

$$= 50 \cdot 93 \text{ ac.}$$

6. See fig. on page 123.

Let $AD = 18$, $CE = 10$; then $AC = \sqrt{10^2 + 9^2} = \sqrt{181}$,

$$\tan ACE = \frac{9}{10} = \cdot 9 = \tan 42^\circ \text{ nearly.}$$

Hence, sector : $\pi r^2 = 2 \times 42^\circ : 360^\circ$;

$$\therefore \text{ sector} = \frac{7}{30} \times \pi \times 18 = 132 \cdot 6.$$

7. As *2s. 6d.* is one-eighth of a pound;

$$\therefore\ 10 \div \frac{1}{8} = 80 \text{ ft.} = \text{area of the alcove,}$$

$$\text{or } \pi r^2 = 160 \text{ ft.}; \ \therefore\ r = \left(\frac{160}{\pi}\right)^{\frac{1}{2}};$$

$$\therefore \text{ length of arc} = \pi r = \sqrt{160\pi} = \sqrt{502{\cdot}656} = 22{\cdot}42 \text{ ft.}$$

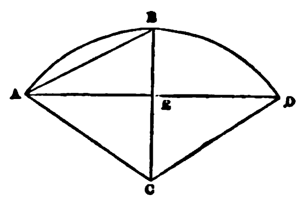
13. By the formula we have

$$\text{length of arc} : 2\pi r :: n : 360,$$

$$\text{and here } n = 29\tfrac{1}{2},\ r = 9;$$

$$\therefore \text{ length of arc} = \frac{59\pi \times 9}{360} = \frac{59\pi}{40} = 4{\cdot}633.$$

14. Let ABD be the arc, C the centre.



Let radius $= r$ and $CE = x$. Then

$$x^2 = -\frac{45^2}{4} + r^2 \ \ldots\ldots\ldots\ldots\ (1),$$

$$\text{and } (r-x)^2 + \frac{45^2}{4} = \frac{51^2}{4} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2);$$

$$\therefore\ (r-x)^2 = \frac{576}{4} \text{ and } r - x = 12 \ \ldots\ldots\ldots\ldots\ (3),$$

$$\text{from (3) and (1)},\ r^2 - 24r + 144 = r^2 - \frac{2025}{4},$$

$$\text{whence } r = \frac{867}{32};$$

$$\therefore\ \sin BCD = \frac{45}{2} \div \frac{867}{32} = \frac{240}{289} = \sin 56{\cdot}14;$$

$$\therefore\ \angle ACB = 112{\cdot}29,$$

$$\text{and length of arc} : 2\pi \times \frac{867}{32} :: 112{\cdot}29 : 360;$$

$$\therefore \text{ length of arc} = \frac{867\pi \times 112{\cdot}29}{360 \times 16} = 53{\cdot}088.$$

Ex. 6.

8.
$$10 : 2\pi r :: 39\tfrac{1}{2} : 360;$$
$$\therefore r = \frac{10 \times 360}{79\pi};$$
$$\therefore \text{area of sector} = \frac{1}{2} \times \frac{10 \times 360}{79\pi} \times 10 = \frac{18000}{79\pi} = 72{\cdot}526 \text{ ft.}$$

9. By the formula in the text, we have
$$20 : 25\pi :: n : 360;$$
$$\therefore n = \frac{4 \times 360}{5\pi} = \frac{288}{\pi} = 91^{\circ}{\cdot}673 = 91^{\circ}\ 40'\ 22''{\cdot}8.$$

10. Area of sector = half the area of triangle = 1210 yds.,

and, length of arc : $2\pi r$ :: 60 : 360;
$$\therefore \text{arc} = \frac{\pi r}{3};$$
$$\therefore \text{area of sector} = \frac{\pi r^2}{6} = 1210,$$
$$\text{whence } r = \left(\frac{7260}{\pi}\right)^{\frac{1}{2}} = 48{\cdot}072 \text{ yds.}$$

Ex. 7.

1. n° being the number of degrees at the centre,
$$\cos\frac{1}{2}n = \frac{12}{42} = \frac{2}{7} = {\cdot}2857143 = \cos 73^{\circ}\ 23'\ 54'';$$
$$\therefore n = 146^{\circ}\ 47'\ 48'' = 146^{\circ}{\cdot}7967;$$
$$\therefore \text{area of sector} : 42^2 \times \pi :: 146{\cdot}7967 : 360;$$
$$\therefore \text{area of sector} = 49\pi \times 14{\cdot}67967 = 2259{\cdot}765 \text{ in.};$$
$$\text{area of triangle} = \frac{1}{2} \times 42^2 \sin 146^{\circ}\ 47'\ 48''$$
$$= 882 \times {\cdot}5476 = 482{\cdot}983 \text{ in.};$$
$$\therefore \text{area of segment} = 2259{\cdot}765 - 482{\cdot}983 = 1776{\cdot}782 \text{ in.}$$

2. If n be the number of degrees at the centre, we have

$$\sin\frac{1}{2}n=\frac{8}{12}=\frac{2}{3}=\cdot 6=\sin 41^\circ\ 48'\ 37'';$$

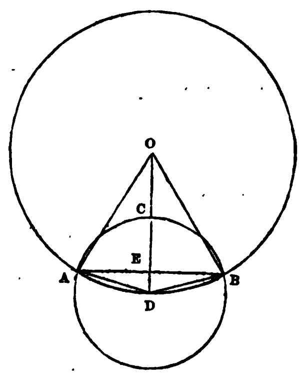
$$\therefore\ n=83^\circ\ 37'\ 14''=83^\circ\cdot 6206;$$

$$\therefore\ \text{area of sector}\ :\ 144\pi\ ::\ 83\cdot 6206\ :\ 360;$$

$$\therefore\ \text{area of sector}=\frac{167\cdot 2412\times\pi}{5}=105\cdot 081;$$

$$\text{area of triangle}=\frac{1}{2}\times 12^2\times\sin 83^\circ\ 37'\ 14''=71\cdot 554;$$

$$\therefore\ \text{area of segment}=105\cdot 081-71\cdot 554=33\cdot 527.$$



10. Let the circles be as in the margin.

It is required to determine the area of the figure $ACBD$. Let $DE=x$; then

$$8^2-x^2=15^2-(15-x)^2$$
$$=30x-x^2;$$

$$\therefore\ x=\frac{64}{30}=\frac{32}{15};$$

$$\therefore\ \cos ADO=\frac{32}{15}\div 8=\frac{4}{15}=\cos 74^\circ\ 32'\ 2''\cdot 36;$$

$$\therefore\ ADB=149^\circ\ 4'\ 5''.$$

Therefore $AOB=2\times(180^\circ-149^\circ\ 4'\ 5'')=61^\circ\ 51'\ 50''$,

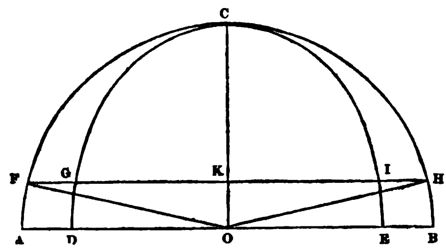
and proceeding as in Ex. 1, we find

$$\text{area of segment}\ ADB=22\cdot 264,$$

$$\text{and area of segment}\ ACB=66\cdot 806,$$

$$\therefore\ \text{area of figure}\ ACBD=89\cdot 07.$$

14. Let DCE be a semi-ellipse having its semi-major axis $OC = 17\frac{1}{2}$ and semi-minor axis $OD = 12\frac{1}{2}$; and let GCI



be the elliptic segment, whose height $(CK) = 10$. Produce DE both ways to A and B, and make $OA = OB = OC = 17\frac{1}{2}$. On AB as diameter describe the circle ACB, and produce GI both ways to meet it in F and H. Join OF, OH.

$$OK = OC - CK = 17\tfrac{1}{2} - 10 = 7\tfrac{1}{2};$$

$$\therefore \cos COH = 7\tfrac{1}{2} \div 17\tfrac{1}{2} = \frac{3}{7} = \cdot 42857 14 = \cos 64^\circ\ 37'\ 23'';$$

$$\therefore\ FOH = 129^\circ\ 14'\ 46'',$$

and, as in Ex. 1, we find

$$\text{area of segment } FCH = 226\cdot 8301,$$

$$\text{and, segment } GCI : \text{segment } FCH :: 25 : 35;$$

$$\therefore \text{ segment } GCI = \frac{5 \times 226\cdot 8301}{7} = 162\cdot 021.$$

Surfaces and Contents of Solids.

Ex. 8.

6. $18 \times 1\frac{1}{2} \times 1\frac{1}{6} = 31\frac{1}{2}$ cub. ft. = the solidity of the log;

$$\therefore 31\tfrac{1}{2} - 2\tfrac{1}{2} = 29 \text{ cub. ft.} = \text{solidity of remaining piece};$$

and therefore we have

$$\text{required length} = \frac{29}{1\frac{1}{2} \times 1\frac{1}{6}} = \frac{29 \times 4}{7} = 16\tfrac{4}{7} \text{ ft.}$$

Ex. 9.

4. Perimeter of either end $= 3 \times 2\frac{1}{4} = \frac{27}{4}$ ft.

$\therefore$ lateral surface $= \frac{101}{4} \times \frac{27}{4} = \frac{2727}{16} = 170{\cdot}4375$ sq. ft.,

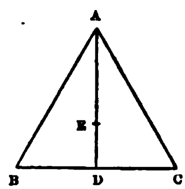
area of the two ends $= 3 \times \frac{9}{8} \times \frac{9}{8} \times \sqrt{3} = \frac{243}{64}\sqrt{3} = 6{\cdot}5764$ sq. ft.;

$\therefore$ whole surface $= 170{\cdot}4375 + 6{\cdot}5764 = 177{\cdot}0139$ sq. ft.

The solid content $= 3{\cdot}2882 \times 25{\cdot}25 = 83{\cdot}027$ cub. ft.

Ex. 10.

1. If the triangle ABC represent the base of the pyramid, and E be the point in which the perpendicular on the plane ABC from the vertex of the pyramid meets the base, it may be easily shewn that the point E will be equidistant from the three angles of the triangle, and



that $DE = \frac{1}{3}AD = \frac{1}{3}\sqrt{\left(\frac{11}{2}\right)^2 - \left(\frac{11}{4}\right)^2} = \frac{(363)^{\frac{1}{2}}}{12}$;

$\therefore$ slant height $= \left(900 + \frac{363}{144}\right)^{\frac{1}{2}} = \frac{(129963)^{\frac{1}{2}}}{12} = 30{\cdot}042$;

$\therefore$ whole surface $= \frac{1}{2} \times 3 \times 5\frac{1}{2} \times 30{\cdot}042 + \frac{1}{2} \times (5\frac{1}{2})^2 . \frac{\sqrt{3}}{2}$

$= 247{\cdot}8465 + 13{\cdot}0987 = 260{\cdot}9452$ sq. ft.

Solid content $= \frac{1}{3} \times 30 \times \frac{1}{2} . (5\frac{1}{2})^2 . \frac{\sqrt{3}}{2}$

$= 10 \times 13{\cdot}098 = 130{\cdot}98$ cub. ft.

7. Here $a^2 = 8 \times \frac{25}{4} \times \cot 22\frac{1}{2} = 50 \times 2{\cdot}414214 = 120{\cdot}7107$,

$b^2 = 8 \times 4 \times \cot 22\frac{1}{2} = 32 \times 2{\cdot}414214 = 77{\cdot}2548$,

and therefore $ab = 10{\cdot}986 \times 8{\cdot}789 = 96{\cdot}556$;

therefore volume

$= \frac{6}{3}(120{\cdot}711 + 96{\cdot}556 + 77{\cdot}255) = 589{\cdot}044$ cub. ft.

Slant height

$$=\left\{6^2+\left(\frac{\sqrt{2}+1}{2}\right)^2\right\}^{\frac{1}{2}}=\sqrt{\frac{147+2\sqrt{2}}{2}}=6{\cdot}1202\,;$$

therefore whole surface

$$=\frac{6{\cdot}1202}{2}\times(32+40)+120{\cdot}7107+77{\cdot}2548$$

$$=418{\cdot}293 \text{ sq. ft.}$$

Ex. 11.

4. Area of either end

$$=\frac{225}{64}\times 3{\cdot}1416=11{\cdot}0447 \text{ sq. ft.}=1{\cdot}2272\,;$$

therefore quantity of earth excavated

$$=1{\cdot}2272\times 15=18{\cdot}408 \text{ cub. yd.}$$

$$\begin{array}{r} 18{\cdot}408 \\ 7\tfrac{1}{4} \\ \hline 128{\cdot}856 \\ 4{\cdot}602 \\ \hline 133{\cdot}458s. \end{array}=£6\ 13{\cdot}458s.=£6\ 13s.\ 5\tfrac{1}{2}d.,$$

therefore cost of excavation £6 13*s.* 5½*d.*

5. Volume of whole globe $=\frac{4}{3}\pi r^3$.

Volume of unknown portion $=\frac{4}{3}\pi(r-5)^3$;

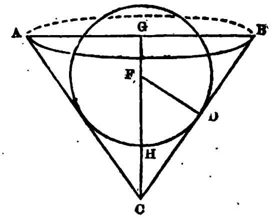
therefore volume of known part

$$=\frac{4}{3}\pi\{r^3-(r-5)^3\}=\frac{4}{3}\pi(15r^2), \text{ nearly};$$

therefore fraction required $=\dfrac{\frac{4}{3}\pi\times 15r^2}{\frac{4}{3}\pi r^3}=\dfrac{15}{r}$

$$=\frac{15}{4000}=\frac{3}{800}, \text{ nearly.}$$

9. Let ABC represent the conical glass; F the centre of the sphere, and FD a radius at right angles to BC the slant side of the cone. If a circle be described about F at the distance FD it will represent the globe *in plano*. H therefore is the lowest point of the sphere; and C, H, and F are in the same straight line, which, if produced will bisect AB in G.



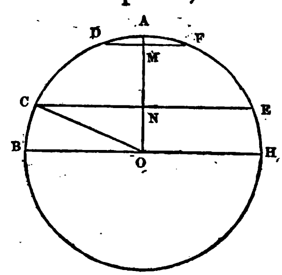
By question, $AG = 2{\cdot}5$ in.; and $GC = 6$ in., and $\therefore AC = 6{\cdot}5$ in., and $AG : AC :: FD : FC$, or $2{\cdot}5 : 6{\cdot}5 :: 2 : FC = 5{\cdot}2$ in.;

$\therefore FG = {\cdot}8$ in., $GH = 2{\cdot}8$ in. = the height of the segment immersed;

therefore solidity of segment, or quantity of water that will run over

$$= \{(4 \times 3) \text{ in.} - (2{\cdot}8 \times 2)\} \text{ in.} \times (2{\cdot}8)^2 \times \frac{\pi}{6} = 26{\cdot}272 \text{ cub. in.}$$

13. Let BAH represent the earth's hemisphere; O its centre, and A the pole. And let DAF, $CDFE$, $BCEH$ represent the three zones.



By question,

$$\text{angle } BOC = AOD = 23^\circ\tfrac{1}{2};$$

$$\text{therefore angle } DOC = 43^\circ,$$

$$AOC = 66^\circ\tfrac{1}{2}.$$

Now $AM = r\,(1 - \cos 23^\circ\frac{1}{2})$ = height of segment DAE;

therefore surface of zone

$$DAE = 2\pi r^2\,(1 - \cos 23^\circ\tfrac{1}{2}).$$

Similarly surface of segment

$$CAE = 2\pi r^2\,(1 - \cos 66^\circ\tfrac{1}{2}),$$

and therefore surface of zone

$$CDFE = 2\pi r^2\,(\cos 23^\circ\tfrac{1}{2} - \cos 66^\circ\tfrac{1}{2}).$$

Similarly surface of zone $BCEH = 2\pi r^2 . \cos 66°\frac{1}{2}$;

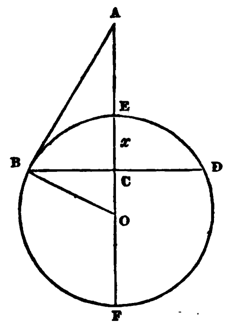
therefore the surfaces of the zones are to each other

as $1 - \cos 23°\frac{1}{2} : \cos 23°\frac{1}{2} - \cos 66°\frac{1}{2} : \cos 66°\frac{1}{2}$,

as $\cdot 083 : \cdot 518 : \cdot 399$, nearly,

as $83 : 518 : 399$, nearly.

15. Let A represent the point above the surface of the earth BED, such that $AE = EO$. Produce AO to meet the circle in E. Let the radius $= r$, and $EC = x$.



Then $AB^2 = AE . AF = 3r^2$;

$\therefore\ BC^2 = 3r^2 - (r + x)^2 = r^2 - (r - x)^2$,

or $2r^2 - 2rx - x^2 = 2rx - x^2$;

$\therefore\ 2rx = r^2$,

$x = \frac{r}{2}$ = height of segment visible;

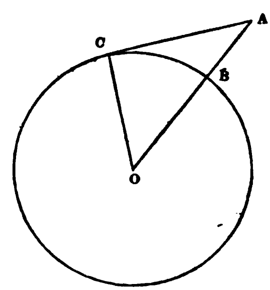
therefore surface of segment $= 2\pi r \times \frac{r}{2} = \pi r^2$;

therefore portion of surface visible $= \pi r^2 \div 4\pi r^2 = \frac{1}{4}$;

also volume of segment $BED = \frac{\pi}{6}(6r - r)\frac{r^2}{4} = \frac{5\pi r^3}{24}$;

therefore portion of volume visible $= \frac{5}{24}\pi r^3 \div \frac{4}{3}\pi r^3 = \frac{5}{32}$.

17. Let B, C represent the two places on the surface of the earth; at C apply the tangent CA; join OB (O being the centre of the circle), and produce it to meet the tangent in A. Join OC. The angle OCA is a right angle. BA is the height to which a man must descend.



$200 : 25000 ::$ angle $COB : 360°$;

$\therefore\ \angle COB = \frac{360}{125} = \frac{72}{25} = 2° 52' 48''$,

$$\text{diameter of the earth} = \frac{25000}{3\cdot1416} = 7957\cdot728 \text{ miles.}$$

$$\text{Now } AO = \frac{OC}{\cos COA} = \frac{3978\cdot864}{\cos 2^\circ\, 52'\, 48''} = \frac{3978\cdot864}{\cdot998737} = 3983\cdot895;$$

therefore required height $= 3983\cdot895 - 3978\cdot864 = 5\cdot031$ miles.

Ex. 13.

7. a^2 = area of mouth $= 3\cdot1416 \times \frac{25}{16} = 4\cdot90875$,

$$\therefore\ a = 2\cdot215;$$

$$b^2 = \text{area of bottom} = 3\cdot1416 \times \frac{1}{4} = \cdot7854,$$

$$\therefore\ b = \cdot886:$$

$$\text{therefore content of glass} = \frac{29}{24} \times (4\cdot9088 + 1\cdot9625 + \cdot7854)$$

$$= \frac{29}{24} \times 7\cdot6567 = 9\cdot252 \text{ cub. in.},$$

and, one imperial gallon $= 277\cdot274$ cub. in.;

$\therefore\ \frac{277\cdot274}{9\cdot252} = 29\cdot969$ = No. of such glasses in an imperial gallon.

9. Since the cone is right angled at the vertex its altitude will be equal to the radius of its base, and therefore equal to the radius of the hemisphere.

$$\text{Volume of the cone} = \frac{1}{3}\pi r^3,$$

$$\text{volume of the hemisphere} = \frac{2}{3}\pi r^3;$$

therefore volume of cone and hemisphere together $= \pi r^3$,

and volume of the cylinder $= 2\pi r^3$.

Volume of additional space $= \pi r^3 = \pi$ cub. ft., since $r = 1$.

ARTIFICERS' WORK.

Ex. 14.

2. $\frac{18 \times 10 \times 2}{3} = 120$ sq. ft. = superficial content of the gable when reduced to standard thickness,

and $\frac{20 \times 30 \times 4}{3} = 800$ sq. ft. = superficial content of the end wall reduced to standard thickness;

therefore whole superficial content $= 800 + 120 = 920$ sq. ft.;

$$\therefore \frac{920}{272{\cdot}25} = 3{\cdot}379 \text{ standard rods of brickwork;}$$

$$\therefore 3{\cdot}379 \times 4 = £13{\cdot}516 = £13\ 10s.\ 4d.$$

5. $21 \times 10 \times 2 + 15 \times 10 \times 2 = 720$ sq. ft. = superficial content of the whole room.

$$6 \times 4 = 24 \text{ ft.} = \text{content of door,}$$

$$2 \times 5 \times 4 = 40 \text{ ft.} = \text{content of windows;}$$

$$\therefore 720 + \frac{40 + 24}{2} = 752 \text{ sq. ft.} = \text{whole amount of work.}$$

As 100 sq. ft. : 752 sq. ft. :: £3·75 : 7·52 × 3·75;
therefore cost of wainscoting = £28·2 = £28 4s.

6. Let x = thickness of lead in inches.

Then area of section of pipe $= \pi\left\{\left(\frac{5}{8} + x\right)^2 - \left(\frac{5}{8}\right)^2\right\}$,

$$\text{content of one yard} = \pi\left(\frac{5}{4}x + x^2\right) \times 36.$$

Hence $36\pi\left(x^2 + \frac{5}{4}x\right) : 1728 = 14 \times 16 : 11325$;

$$\therefore x^2 + \frac{5}{4}x = \frac{48 \times 14 \times 16}{11325\pi}.$$

Whence $x = {\cdot}20737$ in.

MEASUREMENT OF SHOT, SHELLS, AND POWDER.

Ex. 15.

3. Weight of 4 in. ball : weight of 9 in. :: $4^3 : 9^3 :: 1 : \frac{729}{64}$,

weight of 4 in. ball : weight of 6 in. :: $4^3 : 6^3 :: 1 : \frac{216}{64}$;

therefore weight of shell

$$=\frac{9}{64}(729-216)=\frac{9}{64}\times 513 = 72\cdot 14 \text{ lbs.}$$

10. Quantity of powder $=\dfrac{\frac{4}{3}\times 3\cdot 1416\times\frac{729}{8}}{30}=\dfrac{\cdot 31416\times 81}{2}$

$$= 12\cdot 72 \text{ lbs.}$$

13. 10 lbs. = 300 cub. in. = volume of cylinder;

therefore the length $=\dfrac{300}{9\times 3\cdot 1416}=\dfrac{100}{9\cdot 4248}=10\cdot 61$ in.

18. A leaden ball 1 inch in diameter weighs $\frac{24}{7}$ oz.

$$\therefore \frac{24}{7} : 4 :: 1^3 : x^3, \text{ or } x^3=\frac{7}{6};$$

$$\therefore x=\sqrt[3]{\frac{7}{6}}=1\cdot 0527,$$

the diameter of the 4 oz. ball;

therefore calibre of gun $1\cdot 0527+\dfrac{1\cdot 0527}{49}=1\cdot 074$ in.

PLANE TRIGONOMETRY.

TRIGONOMETRICAL FORMULÆ.

Ex. 1.

1. First reducing the minutes and seconds to the decimal parts of a degree,

$$\begin{array}{l} 60\,\underline{)\,17} \\ 60\,\underline{)\,4{\cdot}283} \\ \quad\;{\cdot}0713889 \end{array}$$

$$\therefore\; E = 15{\cdot}0713889$$

$$\frac{E}{9} = \;\;1{\cdot}6745987$$

$$\therefore\; E + \frac{E}{9} = 16{\cdot}7459876$$

or $F = 16^g\; 74'\; 59''{\cdot}876$.

4. $$\text{circular measure} = \frac{57}{180}\cdot\pi = \frac{19 \times 3{\cdot}1416}{60} = {\cdot}99484.$$

6. $$\cos A = {\cdot}6\,;\;\therefore\; \sin A = \sqrt{1 - \cos^2 A} = \sqrt{{\cdot}64} = {\cdot}8 = \frac{4}{5},$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{6}{8} = \frac{3}{4},$$

$$\text{chd}\, A = 2\sin\frac{A}{2} = \sqrt{2\,(1 - \cos A)} = \sqrt{{\cdot}8} = \sqrt{\frac{4}{5}}.$$

9. $$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\tan\frac{A}{2}\,\cos^2\frac{A}{2} = \frac{2\tan\frac{A}{2}}{1 + \tan^2\frac{A}{2}}$$

$$= \frac{2\,(2 - \sqrt{3})}{1 + (7 - 4\sqrt{3})} = \frac{2\,(2 - \sqrt{3})}{4\,(2 - \sqrt{3})} = \frac{1}{2}.$$

Ex. 2.

3. $$\sec A = \frac{\cos \frac{A}{2}}{\cos A \cos \frac{A}{2}} = \frac{\cos\left(A - \frac{A}{2}\right)}{\cos A \cos \frac{A}{2}}$$

$$= \frac{\cos A \cos \frac{A}{2} + \sin A \sin \frac{A}{2}}{\cos A \cos \frac{A}{2}} = 1 + \tan A \tan \frac{A}{2}.$$

4. $$\operatorname{cosec} 2A = \frac{1}{\sin 2A} = \frac{\sin^2 A + \cos^2 A}{2 \sin A \cos A} = \frac{1 + \cot^2 A}{2 \cot A}.$$

11. $$\sec^2 A . \operatorname{cosec}^2 A = \sec^2 A (1 + \cot^2 A)$$

$$= \sec^2 A + \frac{1}{\cos^2 A} . \frac{\cos^2 A}{\sin^2 A}$$

$$= \sec^2 A + \frac{1}{\sin^2 A} = \sec^2 A + \operatorname{cosec}^2 A.$$

12. $$\frac{\cos^2 A - \sin^2 B}{\sin^2 A \sin^2 B} = \cot^2 A \operatorname{cosec}^2 B - \operatorname{cosec}^2 A$$

$$= \cot^2 A (1 + \cot^2 B) - (1 + \cot^2 A)$$

$$= \cot^2 A . \cot^2 B - 1.$$

13. $$\tan 2A - \tan A = \frac{\sin 2A}{\cos 2A} - \frac{\sin A}{\cos A}$$

$$= \frac{\sin (2A - A)}{\cos 2A \cos A}$$

$$= \frac{2 \sin A}{\cos A + \cos 3A}.$$

15. $$\frac{\sin 2A}{1 + \cos 2A} \times \frac{\cos A}{1 + \cos A} = \tan A \times \frac{\cos A}{1 + \cos A}$$

$$= \frac{\sin A}{1 + \cos A} = \tan \frac{A}{2}.$$

18. $\sin 5A \sin A = \frac{1}{2}(\cos 4A - \cos 6A)$

$$= \frac{1}{2}\{1 - 2\sin^2 2A - (1 - 2\sin^2 3A)\}$$

$$= \sin^2 3A - \sin^2 2A.$$

21. $\tan 3A \,.\, \tan A = \dfrac{\sin 3A \,.\, \sin A}{\cos 3A \,.\, \cos A}$

$$= \frac{\frac{1}{2}\{\cos(3A - A) - \cos(3A + A)\}}{\frac{1}{2}\{\cos(3A - A) + \cos(3A + A)\}}$$

$$= \frac{\cos 2A - \cos 4A}{\cos 2A + \cos 4A}.$$

32. $\tan(A + B) = \dfrac{\sin(A+B)}{\cos(A+B)} = \dfrac{\sin(A+B)\sin(A-B)}{\cos(A+B)\sin(A-B)}$

$$= \frac{\sin^2 A - \sin^2 B}{\frac{1}{2}(\sin 2A - \sin 2B)} = \frac{\sin^2 A - \sin^2 B}{\sin A \,.\, \cos A - \sin B \,.\, \cos B}.$$

34. $\sin(A - B)\sin C + \sin(B - C)\sin A - \sin(A - C)\sin B$

$$= \frac{1}{2}\{\cos(A - B - C) - \cos(A - B + C) + \cos(B - C - A)$$
$$- \cos(B - C + A) + \cos(C - A - B) - \cos(C - A + B)\}$$

$= 0$, since the first and sixth, second and third, fourth and fifth terms destroy each other.

40. $\sin A + \sin B + \sin C - \sin(A + B + C)$

$= (\sin A + \sin B) - \{\sin(A + B + C) - \sin C\}$

$= 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) - 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A+B+2C)$

$= 2\sin\dfrac{A+B}{2}\left\{\cos\dfrac{A-B}{2} - \cos\dfrac{A+B+2C}{2}\right\}$

$= 2\sin\dfrac{A+B}{2}\left\{2\sin\dfrac{A+C}{2}\sin\dfrac{B+C}{2}\right\};$

$\therefore \sin A + \sin B + \sin C$

$$= 4\sin\frac{A+B}{2}\sin\frac{A+C}{2}\sin\frac{B+C}{2} + \sin(A + B + C).$$

42. $\tan A+\tan B+\tan C=\dfrac{\sin C}{\cos C}+\dfrac{\sin(A+B)}{\cos A\cos B}$

$$=\frac{\{\cos A\cos B-\cos(A+B)\}\sin C+\sin(A+B)\cos C+\cos(A+B)\sin C}{\cos A\,.\cos B\,.\cos C}$$

$$=\frac{\sin A\sin B\,.\sin C+\sin(A+B+C)}{\cos A\,.\cos B\,.\cos C}$$

$$=\tan A\,.\tan B\,.\tan C+\frac{\sin(A+B+C)}{\cos A\,.\cos B\,.\cos C}.$$

44. $2\sin\dfrac{A+B+C}{2}\sin\dfrac{B+C-A}{2}=\cos A-\cos(B+C)$

$$2\sin\frac{A+C-B}{2}\sin\frac{A+B-C}{2}=\cos(B-C)-\cos A;$$

$$\therefore 4\sin\frac{1}{2}(A+B+C)\sin\frac{1}{2}(B+C-A)\sin\frac{1}{2}(A+C-B)\sin\frac{1}{2}(A+B-C)$$

$$=-\cos^2A-\cos(B+C)\cos(B-C)+\cos A\{\cos(B+C)+\cos(B-C)\}$$

$$=1-\cos^2A-\cos^2B-\cos^2C+2\cos A\cos B\cos C.$$

48. $\dfrac{\sin A}{\sin B\cos B}-\dfrac{\sin A}{\sin B}=\dfrac{\sin A}{\sin B}\cdot\dfrac{1-\cos B}{\cos B}=\dfrac{\sin A\times 2\sin^2\dfrac{B}{2}}{\sin B\cos B}$,

$$1-\frac{\cos(A-B)}{\cos B}=\frac{\cos B-\cos(A-B)}{\cos B}=\frac{2\sin\dfrac{A}{2}\sin\left(\dfrac{A}{2}-B\right)}{\cos B}$$

$$=2\sin\frac{A}{2}-\sin A\frac{\sin B}{\cos B};$$

$$\therefore\frac{\sin A}{\sin B\cos B}-\frac{\cos(A-B)}{\cos B}-\frac{\sin A}{\sin B}+1$$

$$=2\sin^2\frac{A}{2}-\frac{\sin A}{\sin B}\left(\frac{\sin^2B}{\cos B}-\frac{2\sin^2\dfrac{B}{2}}{\cos B}\right)$$

$$=2\sin^2\frac{A}{2}-2\sin^2\frac{B}{2}\cdot\frac{\sin A}{\sin B}\left(\frac{2\cos^2\frac{1}{2}B-1}{\cos B}\right)$$

$$=\frac{2\left(\sin B\sin^2\frac{1}{2}A-\sin A\sin^2\frac{1}{2}B\right)}{\sin B}.$$

Ex. 3.

1. $$\sin 7^\circ 30' = \sqrt{\frac{1-\cos 15^\circ}{2}} = \sqrt{\frac{1}{2}\left(1-\sqrt{\frac{1+\cos 30^\circ}{2}}\right)}$$
$$= \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{3}}}.$$

2. $$\cos 12^\circ = \cos(30^\circ - 18^\circ) = \cos 30^\circ \cos 18^\circ + \sin 30^\circ \sin 18^\circ$$
$$= \frac{\sqrt{3}}{2} \times \frac{1}{4}\sqrt{10+2\sqrt{5}} + \frac{1}{2} \times \frac{1}{4}(\sqrt{5}-1) = \frac{1}{8}\{\sqrt{5}-1+(30+6\sqrt{5})^{\frac{1}{2}}\}.$$

3. $$\tan 37^\circ 30' = \frac{1-\cos 75^\circ}{\sin 75^\circ} = \frac{1-\sin 15^\circ}{\cos 15^\circ}$$
$$= \sqrt{6}-\sqrt{2}-(2-\sqrt{3}).$$

Ex. 4.

1. $$\sin 3A = 3\sin A - 4\sin^3 A = 4\sin A\left(\frac{3}{4}-\sin^2 A\right)$$
$$= 4\sin A(\sin^2 60^\circ - \sin^2 A)$$
$$= 4\sin A \sin(60^\circ + A)\sin(60^\circ - A).$$

2. $$\sec(45^\circ + A)\sec(45^\circ - A)$$
$$= \frac{1}{\cos(45^\circ + A)\cos(45^\circ - A)} = \frac{1}{\cos^2 45^\circ - \sin^2 A}$$
$$= \frac{1}{\frac{1}{2}-\sin^2 A} = \frac{2}{1-2\sin^2 A} = \frac{2}{\cos 2A} = 2\sec 2A.$$

3. $$\tan\left(45^\circ+\frac{A}{2}\right)+\cot\left(45^\circ+\frac{A}{2}\right) = \frac{\tan^2\left(45^\circ+\frac{A}{2}\right)+1}{\tan\left(45^\circ+\frac{A}{2}\right)}$$
$$= \frac{\sec^2\left(45^\circ+\frac{A}{2}\right)}{\tan\left(45^\circ+\frac{A}{2}\right)} = \frac{1}{\sin\left(45^\circ+\frac{A}{2}\right)\cos\left(45^\circ+\frac{A}{2}\right)}$$
$$= \frac{2}{\sin(90^\circ+A)} = \frac{2}{\cos A} = 2\sec A.$$

7. $\text{chd } 108° = 2 \sin 54° = 2 \times \frac{1}{4}(\sqrt{5}+1)$

$$= 2\left\{\frac{1}{4}(\sqrt{5}-1)+\frac{1}{2}\right\} = 2\sin 18° + 2\sin 30°$$

$$= \text{chd } 36° + \text{chd } 60°.$$

Ex. 5.

1. $\tan(A+B+C) = \dfrac{\tan A + \tan(B+C)}{1-\tan A \tan(B+C)}$

$$= \frac{\tan A + \dfrac{\tan B + \tan C}{1-\tan B\tan C}}{1 - \dfrac{\tan A(\tan B+\tan C)}{1-\tan B\tan C}}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan A\tan C - \tan B\tan C};$$

but since $A+B+C = 90$;

$\therefore \tan(A+B+C) = \infty$, and therefore

$1 - \tan A\tan B + \tan A\tan C - \tan B\tan C = 0$,

or $\tan A\tan B + \tan A\tan C + \tan B\tan C = 1$.

2. $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot B + \cot A}$;

$$\therefore \cot(A+B+C) = \frac{\cot(A+B)\cot C - 1}{\cot C + \cot(A+B)} = 0;$$

$$\therefore \cot(A+B)\cot C = 1,$$

or $\dfrac{\cot A\cot B - 1}{\cot B + \cot A}\cot C = 1$;

whence $\cot A + \cot B + \cot C = \cot A\cot B\cot C$.

5. $\cos A + \sin C - \sin B = \sin(B+C) + \sin C - \sin B$

$$= 2\cos\frac{B+C}{2}\left(\sin\frac{B+C}{2} - \sin\frac{B-C}{2}\right)$$

$$= 4\cos\frac{B+C}{2}\sin\frac{C}{2}\cos\frac{B}{2}.$$

Similarly, $\cos B + \sin C - \sin A = 4 \cos \frac{A+C}{2} \sin \frac{C}{2} \cos \frac{A}{2}$;

$$\therefore \frac{\cos A + \sin C - \sin B}{\cos B + \sin C - \sin A} = \frac{\cos \frac{B+C}{2} \cos \frac{B}{2}}{\cos \frac{A+C}{2} \cos \frac{A}{2}}$$

$$= \frac{\cos\left(45^\circ - \frac{A}{2}\right) \cos \frac{B}{2}}{\cos \frac{A}{2} \cos\left(45^\circ - \frac{B}{2}\right)} = \frac{1 + \tan \frac{A}{2}}{1 + \tan \frac{B}{2}}.$$

Ex. 6.

1. Since $A + B + C = 180$;

$$\therefore \sin 2C = \sin \{360^\circ - 2(A+B)\} = -\sin 2(A+B)$$

$$(\sin 2A + \sin 2B) + \sin 2C = 2 \sin (A+B) \cos (A-B)$$
$$- 2 \sin (A+B) \cos (A+B)$$
$$= 2 \sin (A+B) \{\cos (A-B) - \cos (A+B)\}$$
$$= 2 \sin C \cdot 2 \sin A \sin B$$
$$= 4 \sin A \sin B \sin C.$$

3. Since $C = 180^\circ - (A+B)$; $\therefore \cos C = -\cos (A+B)$;

$$\therefore \cos^2 C + 2 \cos A \cos B \cos C$$
$$= -\cos (A+B) \{-\cos (A+B) + 2 \cos A \cos B\}$$
$$= -\cos (A+B) \cos (A-B)$$
$$= -\frac{\cos 2A + \cos 2B}{2}.$$

Hence $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C$

$$= \frac{1 + \cos 2A}{2} + \frac{1 + \cos 2B}{2} - \frac{\cos 2A + \cos 2B}{2} = 1.$$

4. $\sin C = \sin(A+B)$;

$$\therefore (\sin A + \sin B) + \sin C$$
$$= 2\sin\frac{A+B}{2}\left\{\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right\}$$
$$= 2\cos\frac{C}{2} \times 2\cos\frac{A}{2}\cos\frac{B}{2}$$
$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

10. If $A' = A-\theta,\ B' = B-\theta,\ C' = C-\theta,$

$$4\sin A'\sin B'\sin C' = 2\sin C'\{\cos(A'-B') - \cos(A'+B')\}$$
$$= \sin(C'+B'-A') + \sin(A'+C'-B')$$
$$+ \sin(A'+B'-C') - \sin(A'+B'+C').$$

Now $4\sin^3\theta = 4\sin(A-\theta)\sin(B-\theta)\sin(C-\theta)$;

$$\therefore 4\sin^3\theta = \sin(A+B-C-\theta) + \sin(A+C-B-\theta)$$
$$+ \sin(B+C-A-\theta) - \sin(A+B+C-3\theta)$$
$$= \sin(2C+\theta) + \sin(2B+\theta) + \sin(2A+\theta) - \sin 3\theta,$$

or $3\sin\theta = \sin(2C+\theta) + \sin(2B+\theta) + \sin(2A+\theta)$,

hence,

$$(\sin 2A + \sin 2B + \sin 2C)\cot\theta = 3 - (\cos 2A + \cos 2B + \cos 2C);$$
$$\therefore 4\sin A\sin B\sin C\cot\theta = 4(1 + \cos A\cos B\cos C),$$
$$\text{or } \cot\theta = \frac{1 + \cos A\cos B\cos C}{\sin A\sin B\sin C}$$
$$= \cot A + \cot B + \cot C \text{ by Ex. } 2_{43}.$$

Also $\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \cot^2 A + \cot^2 B + \cot^2 C$

$$+ 2(\cot A\cot B + \cot A\cot C + \cot B\cot C)$$
$$= 1 + \cot^2 A + \cot^2 B + \cot^2 C + 2,$$

(since $\cot A\cot B + \cot A\cot C + \cot B\cot C = 1$),

hence $\operatorname{cosec}^2\theta = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C.$

Ex. 7.

1. Here, $\sin A - \sin B = \sin B - \sin C$,

$$\text{or,}\quad \sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B) = \sin\frac{1}{2}(B-C)\cos\frac{1}{2}(B+C);$$

$$\therefore\ \sin\frac{1}{2}(A-B)\sin\frac{1}{2}C = \sin\frac{1}{2}(B-C)\sin\frac{A}{2},$$

$$\text{or,}\quad \sin\frac{A-B}{2} \div \sin\frac{A}{2}\sin\frac{B}{2} = \sin\frac{B-C}{2} \div \sin\frac{B}{2}\sin\frac{C}{2},$$

$$\text{that is,}\quad \cot\frac{B}{2} - \cot\frac{A}{2} = \cot\frac{C}{2} - \cot\frac{B}{2};$$

whence $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ are also in Arith. Prog.

3. $\cos(A-C)\cos B = \cos(A+C-B)$

$$= \cos(A+C)\cos B + \sin(A+C)\sin B;$$

$$\therefore\ \{\cos(A-C) - \cos(A+C)\}\cos B$$

$$= 2\sin A\sin C\cos B = \sin(A+C)\sin B,$$

$$\text{or}\quad 2\cot B = \frac{\sin(A+C)}{\sin A\sin C} = \cot A + \cot C;$$

$\therefore$ $\cot A$, $\cot B$, $\cot C$ are in Arith. Prog.

and $\therefore$ $\tan A$, $\tan B$, $\tan C$ are in Harmonic Prog.

7. $\dfrac{\cos B}{\cos A} = \dfrac{1}{\cos C}$; $\therefore$ $\dfrac{\cos B - \cos A}{\cos B + \cos A} = \dfrac{1-\cos C}{1+\cos C}$,

$$\frac{2\sin\frac{A+B}{2}\sin\frac{A-B}{2}}{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}} = \frac{2\sin^2\frac{C}{2}}{2\cos^2\frac{C}{2}};$$

$$\therefore\ \tan\frac{1}{2}(A+B)\tan\frac{1}{2}(A-B) = \tan^2\frac{C}{2}.$$

Ex. 8.

1.
$$\sin A = \sin 2A = 2 \sin A \cos A;$$
$$\therefore \cos A = \frac{1}{2} = \cos 60°,$$
$$\text{and } \sin A = 0;$$
$$\therefore A = 0; \text{ or } 60°.$$

6.
$$\sin^2 2A - \sin^2 A = \frac{1}{4};$$
$$\therefore 4 \sin^2 A (1 - \sin^2 A) - \sin^2 A = \frac{1}{4};$$
$$\sin^4 A - \frac{3}{4} \sin^2 A + \frac{9}{64} = \frac{9}{64} - \frac{1}{16} = \frac{5}{64};$$
$$\therefore \sin^2 A = \frac{3 \pm \sqrt{5}}{8} = \frac{6 \pm 2\sqrt{5}}{16} = \frac{5 \pm 2\sqrt{5} + 1}{16};$$
$$\therefore \sin A = \pm \frac{\sqrt{5} \pm 1}{4} = \sin 54°, \sin (180° \pm 54°);$$
$$\sin 18°, \sin (180° \pm 18°).$$

9.
$$\tan A + \cot A = 4;$$
$$\therefore \tan^2 A - 4 \tan A + 4 = 4 - 1 = 3;$$
$$\therefore \tan A = 2 \pm \sqrt{3} = \frac{4 \pm 2\sqrt{3}}{2} = \frac{(\sqrt{3} \pm 1)^2}{(3 - 1)} = \frac{\sqrt{3} \pm 1}{\sqrt{3} \mp 1}$$
$$= \frac{1 \pm \frac{1}{\sqrt{3}}}{1 \mp \frac{1}{\sqrt{3}}} = \frac{\tan 45° \pm \tan 30°}{1 \mp \tan 45° \tan 30°} = \tan (45° \pm 30°);$$
$$\therefore A = 75°, \text{ or } 15°.$$

11. $$(\cos A + \cos 3A) + \cos 2A = 2 \cos A \cos 2A + \cos 2A = 0;$$
$$\therefore (1) \ \cos 2A = 0, \quad \text{or } 2A = 90°, \ A = 45°,$$
$$\text{and } (2) \quad \cos A = -\frac{1}{2}, \text{ or } A = 120°.$$

Ex. 9.

1. $$\sin(x+a) = \cos\{90^\circ - (x+a)\} = \cos(x-a);$$
$$\therefore\ x - a = 90^\circ - x - a;$$
$$\therefore\ 2x = 90^\circ,$$
$$x = 45^\circ.$$

2. $$\sqrt{2}(\cos 3x + \sin 3x) = 1;$$
$$\therefore\ \frac{1}{\sqrt{2}}\cos 3x + \frac{1}{\sqrt{2}}\sin 3x = \frac{1}{2};$$
$$\cos(3x - 45^\circ) = \cos 60^\circ;$$
$$\therefore\ x = \frac{45^\circ + 60^\circ}{3} = 35^\circ.$$

6. $$\tan\alpha . \tan x = \{\tan(\alpha+x) + \tan(\alpha-x)\} \times \{\tan(\alpha+x) - \tan(\alpha-x)\},$$
or $$\frac{\sin\alpha . \sin x}{\cos\alpha . \cos x} = \frac{\sin\{(\alpha+x)+(\alpha-x)\}}{\cos(\alpha+x)\cos(\alpha-x)} . \frac{\sin\{(\alpha+x)-(\alpha-x)\}}{\cos(\alpha+x)\cos(\alpha-x)}$$
$$= \frac{\sin 2\alpha \sin 2x}{\cos^2(\alpha+x)\cos^2(\alpha-x)},$$
hence $\sin x = 0$, and $\cos^2(\alpha+x)\cos^2(\alpha-x) = 4\cos^2\alpha\cos^2 x$,
$$\text{or } \cos(\alpha+x)\cos(\alpha-x) = \pm 2\cos\alpha . \cos x;$$
$$\therefore\ \cos^2 x - \sin^2\alpha = \pm 2\cos\alpha\cos x,$$
$$\text{and } \cos^2 x \pm 2\cos\alpha\cos x + \cos^2\alpha = 1;$$
$$\therefore\ \cos x = \pm(1 \pm \cos\alpha).$$

9. $$\tan^3 x = \tan(x-a); \quad \therefore\ \tan^2 x = \frac{\tan(x-a)}{\tan x};$$
$$\therefore\ \frac{1-\tan^2 x}{1+\tan^2 x} = \frac{\tan x - \tan(x-a)}{\tan x + \tan(x-a)};$$
$$\therefore\ \cos 2x = \frac{\sin a}{\sin(2x-a)};$$

$$\therefore\ 2\sin\alpha = 2\sin(2x-\alpha)\cos 2x = \sin(4x-\alpha) - \sin\alpha;$$

$$\therefore\ \sin(4x-\alpha) = 3\sin\alpha;$$

$$\therefore\ x = \frac{1}{4}\{\alpha + \sin^{-1}(3\sin\alpha)\}.$$

Ex. 10.

3. $$\text{Let } A = \tan^{-1}\frac{1}{7},\quad \text{then } \tan A = \frac{1}{7},$$

$$B = \tan^{-1}\frac{1}{3},\ \text{then } \tan B = \frac{1}{3},\ \text{and } \tan 2B = \frac{2\times\frac{1}{3}}{1-\frac{1}{9}} = \frac{3}{4},$$

$$\tan(A+2B) = \frac{\tan A + \tan 2B}{1-\tan A\tan 2B} = \frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\times\frac{3}{4}} = 1;$$

$$\therefore\ A + 2B = \tan^{-1}1,\ \text{or } \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3} = 45^\circ.$$

5. $$\text{Let } \theta = \sin^{-1}\frac{1}{\sqrt{5}},\ \text{then } \sin\theta = \frac{1}{\sqrt{5}},\ \operatorname{cosec}^2\theta = 5;$$

$$\therefore\ \cot\theta = 2;\quad \phi = \cot^{-1}3,\ \text{then } \cot\phi = 3,$$

$$\cot(\theta+\phi) = \frac{\cot\theta\cot\phi - 1}{\cot\phi + \cot\theta} = \frac{6-1}{3+2} = 1 = \cot 45^\circ;$$

$$\therefore\ \theta+\phi \text{ or } \sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3 = 45^\circ.$$

8. $$\tan^{-1}t_1 - \tan^{-1}t_2 = \tan^{-1}\frac{t_1-t_2}{1+t_1t_2},$$

$$\tan^{-1}t_2 - \tan^{-1}t_3 = \tan^{-1}\frac{t_2-t_3}{1+t_2t_3},$$

$$\cdots\cdots\cdots\cdots = \cdots\cdots\cdots\cdots$$

$$\tan^{-1}t_{n-1} - \tan^{-1}t_n = \tan^{-1}\frac{t_{n-1}-t_n}{1+t_{n-1}t_n};$$

$$\therefore \tan^{-1} t_1 - \tan^{-1} t_n = \tan^{-1} \frac{t_1 - t_2}{1 + t_1 t_2} + \tan^{-1} \frac{t_2 - t_3}{1 + t_2 t_3} + \&c.$$
$$+ \tan^{-1} \frac{t_{n-1} - t_n}{1 + t_{n-1} t_n}.$$

10. $$\tan^{-1} \frac{x \cos\phi}{1 - x \sin\phi} - \tan^{-1} \frac{x - \sin\phi}{\cos\phi}$$
$$= \tan^{-1} \frac{\dfrac{x\cos\phi}{1 - x\sin\phi} - \dfrac{x - \sin\phi}{\cos\phi}}{1 + \dfrac{x\cos\phi}{1 - x\sin\phi} \cdot \dfrac{x - \sin\phi}{\cos\phi}}$$
$$= \tan^{-1}(\tan\phi) = \phi.$$

Ex. 11.

1. $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{\sqrt{3}}; \quad \text{and} \quad \tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{1}{\sqrt{15}};$$
$$\therefore \frac{\sin\theta}{\cos\theta} + \frac{\sin\phi}{\cos\phi} = \frac{\sin(\theta+\phi)}{\cos\theta\cos\phi} = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{15}},$$
$$\text{or} \quad \frac{\sin(\theta+\phi)}{\dfrac{\sqrt{3}}{2} \times \dfrac{\sqrt{15}}{2}} = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{15}};$$
$$\therefore \sin(\phi+\theta) = \frac{\sqrt{15}+\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \times \frac{\sqrt{5}+1}{2} = \sin 60^\circ \cos 36^\circ.$$

2. Let $\theta = \cot^{-1}(x-1)$, then $\cot\theta = x - 1$,
$$\phi = \cot^{-1}(x+1), \qquad \cot\phi = x + 1,$$
$$\cot(\theta - \phi) = \frac{\cot\theta\cot\phi + 1}{\cot\phi - \cot\theta} = \frac{x^2 - 1 + 1}{x + 1 - (x-1)} = \frac{x^2}{2},$$
$$\cot\frac{\pi}{12} = \cot 15^\circ = \frac{1 + \cos 30^\circ}{\sin 30^\circ} = \frac{2 + \sqrt{3}}{1};$$
$$\therefore x^2 = 4 + 2\sqrt{3}; \quad \therefore x = \pm(\sqrt{3} + 1).$$

5. $$\cos v = \frac{\cos u - e}{1 - e\cos u};$$

$$\therefore \frac{1-\cos v}{1+\cos v} = \frac{1 - e\cos u - (\cos u - e)}{1 - e\cos u + \cos u - e} = \frac{(1+e)(1-\cos u)}{(1-e)(1+\cos u)},$$

and $\tan^2\dfrac{v}{2} = \dfrac{1+e}{1-e}\tan^2\dfrac{u}{2}$, or $\tan\dfrac{v}{2} = \left(\dfrac{1+e}{1-e}\right)^{\frac{1}{2}}\tan\dfrac{u}{2}$.

6. $$a\cos 2\theta + b\sin 2\theta = a(\cos 2\theta + \tan\theta\sin 2\theta)$$
$$= \frac{a}{\cos\theta}\cos(2\theta - \theta)$$
$$= a.$$

7. $$1 + \tan^2\theta = \frac{a^{\frac{2}{3}} + b^{\frac{2}{3}}}{a^{\frac{2}{3}}}; \quad \therefore \frac{a}{\cos\theta} = a^{\frac{2}{3}}\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}},$$

$$1 + \cot^2\theta = \frac{a^{\frac{2}{3}} + b^{\frac{2}{3}}}{b^{\frac{2}{3}}}; \quad \therefore \frac{b}{\sin\theta} = b^{\frac{2}{3}}\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}};$$

$$\therefore \frac{a}{\cos\theta} + \frac{b}{\sin\theta} = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}.$$

10. $$\cos\phi = \frac{1 - \tan^2\dfrac{\phi}{2}}{1 + \tan^2\dfrac{\phi}{2}} = \frac{1 - \tan^{\frac{2}{3}}\theta}{1 + \tan^{\frac{2}{3}}\theta} = \frac{\cos^{\frac{2}{3}}\theta - \sin^{\frac{2}{3}}\theta}{\cos^{\frac{2}{3}}\theta + \sin^{\frac{2}{3}}\theta},$$

and $$m^2 = 1 + 3\cos^2\phi = 1 + 3\left\{\frac{\cos^{\frac{4}{3}}\theta - 2\sin^{\frac{2}{3}}\theta\cos^{\frac{2}{3}}\theta + \sin^{\frac{4}{3}}\theta}{\cos^{\frac{4}{3}}\theta + 2\sin^{\frac{2}{3}}\theta\cos^{\frac{2}{3}}\theta + \sin^{\frac{4}{3}}\theta}\right\}$$

$$= 4\left\{\frac{\cos^{\frac{4}{3}}\theta - \sin^{\frac{2}{3}}\theta\cos^{\frac{2}{3}}\theta + \sin^{\frac{4}{3}}\theta}{\left(\cos^{\frac{2}{3}}\theta + \sin^{\frac{2}{3}}\theta\right)^2}\right\} = 4\left\{\frac{\cos^2\theta + \sin^2\theta}{\left(\cos^{\frac{2}{3}}\theta + \sin^{\frac{2}{3}}\theta\right)^3}\right\}$$

$$= \frac{4}{\left(\cos^{\frac{2}{3}}\theta + \sin^{\frac{2}{3}}\theta\right)^3}; \text{ hence } m = \frac{2}{\left(\cos^{\frac{2}{3}}\theta + \sin^{\frac{2}{3}}\theta\right)^{\frac{3}{2}}},$$

$$\text{and } \therefore \frac{1}{m} = \frac{1}{2}\left(\cos^{\frac{2}{3}}\theta + \sin^{\frac{2}{3}}\theta\right)^{\frac{3}{2}}.$$

11. $$\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}=\tan\frac{\beta}{2}\left(1+\tan^2\frac{\beta}{2}\right),$$

$$1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=1-\tan^4\frac{\beta}{2};$$

$$\therefore\ \tan\frac{\alpha+\beta}{2}=\frac{\tan\frac{\beta}{2}\left(1+\tan^2\frac{\beta}{2}\right)}{1-\tan^4\frac{\beta}{2}}=\frac{\tan\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}}=\frac{1}{2}\tan\beta=\tan\phi;$$

$$\therefore\ 2\phi=\alpha+\beta.$$

13. $$1-\cos\theta=1-\cos^2\alpha+\sin^2\alpha\,(1-c^2\sin^2\theta)^{\frac{1}{2}}$$

$$=\sin^2\alpha\,\{1+(1-c^2\sin^2\theta)^{\frac{1}{2}}\}=\frac{c^2\sin^2\alpha\sin^2\theta}{1-(1-c^2\sin^2\theta)^{\frac{1}{2}}};$$

$$\therefore\ 1-(1-c^2\sin^2\theta)^{\frac{1}{2}}=\frac{c^2\sin^2\alpha\,(1-\cos^2\theta)}{1-\cos\theta}=c^2\sin^2\alpha\,(1+\cos\theta).$$

Multiplying by $\sin^2\alpha$ and substituting for the radical term, we get

$$\sin^2\alpha-(\cos^2\alpha-\cos\theta)=c^2\sin^4\alpha+c^2\sin^4\alpha\cos\theta;$$

$$\therefore\ \cos\theta=\frac{1-2\sin^2\alpha+c^2\sin^4\alpha}{1-c^2\sin^4\alpha},$$

Dividendo and Componendo,

$$\frac{1-\cos\theta}{1+\cos\theta}=\frac{2\sin^2\alpha-2c^2\sin^4\alpha}{2-2\sin^2\alpha};$$

$$\therefore\ \tan^2\frac{\theta}{2}=\tan^2\alpha\,(1-c^2\sin^2\alpha).$$

Ex. 12.

2. $$\frac{\tan(\theta+\phi)}{\tan(\theta-\phi)}=\frac{a+b}{a-b};$$

$$\therefore\ \frac{\tan(\theta+\phi)+\tan(\theta-\phi)}{\tan(\theta+\phi)-\tan(\theta-\phi)}\ \text{or}\ \frac{\sin 2\theta}{\sin 2\phi}=\frac{a}{b}.$$

Hence $b^2\sin^2 2\theta=a^2\sin^2 2\phi$.

But $b^2\cos^2 2\theta=a^2\cos^2 2\phi-2ac\cos 2\phi+c^2$;

$$\therefore\ \cos 2\phi=\frac{a^2+c^2-b^2}{2ac}.$$

4. $a\tan^2\theta + b = m(1+\tan^2\theta)$; $\therefore\ (m-a)\tan^2\theta = b-m$,

$b\tan^2\phi + a = n(1+\tan^2\phi)$; $\therefore\ (n-b)\tan^2\phi = a-n$;

$$\therefore\ \frac{m-a}{n-b}\times\frac{b^2}{a^2}=\frac{b-m}{a-n};$$

$$\therefore\ (m-b)(n-b)a^2 = (m-a)(n-a)b^2;$$

$$\therefore\ \{mn-(m+n)b+b^2\}a^2 = \{mn-(m+n)a+a^2\}b^2;$$

$$\therefore\ mn(a^2-b^2)-(m+n)(a-b)ab = 0;$$

$$\therefore\ \frac{mn}{m+n}=\frac{ab}{a+b}.$$

6. From the equations, (2) and (3),

$$b^2\cos^2\omega + a^2\sin^2\omega = \frac{a^2b^2}{R^2} = \frac{V^2}{\mu}\sin^2\alpha,$$

whence $(a^2+b^2)-(a^2-b^2)\cos 2\omega = \dfrac{V^2}{\mu}(1-\cos 2\alpha)$;

$\therefore$ from (1), $(a^2-b^2)\cos 2\omega = \dfrac{V^2}{\mu}\cos 2\alpha + R^2$ ....(4).

Again, from (1) and (2),

$$a+b=\sqrt{\frac{V^2}{\mu}+R^2+\frac{VR}{\mu^{\frac{1}{2}}}\times 2\sin\alpha},$$

$$\text{and } a-b=\sqrt{\frac{V^2}{\mu}+R^2-\frac{2VR}{\mu^{\frac{1}{2}}}\sin\alpha};$$

$$\therefore\ a^2-b^2=\left\{\left(\frac{V^2}{\mu}+R^2\right)^2-\frac{2V^2R^2}{\mu}\times 2\sin^2\alpha\right\}^{\frac{1}{2}}$$

$$=\left\{\frac{V^4}{\mu^2}+R^4+\frac{2V^2R^2}{\mu}\cos 2\alpha\right\}^{\frac{1}{2}} \text{ ....(5)},$$

from (4) and (5),

$$\cos^2 2\omega = \frac{\dfrac{V^4}{\mu^2}\cos^2 2\alpha + \dfrac{2V^2R^2}{\mu}\cos 2\alpha + R^4}{\dfrac{V^4}{\mu^2}+\dfrac{2V^2R^2}{\mu}\cos 2\alpha + R^4};$$

$$\therefore \sin^2 2\omega = \frac{V^4}{\mu^2}\sin^2 a \div \left(\frac{V^4}{\mu^2} + \frac{2V^2R^2}{\mu}\cos 2a + R^4\right);$$

$$\therefore \cot 2\omega = \left(\frac{V^2}{\mu}\cos 2a + R^2\right) \div \frac{V^2}{\mu}\sin 2a = \cot 2a + \frac{\mu R^2}{V^2}\operatorname{cosec} 2a.$$

Properties of Plane Figures.

Ex. 13.

2. $$\cos\{2A - B\} = \cos\left\{2A - \left(\frac{\pi}{2} - A\right)\right\} = \cos\left(3A - \frac{\pi}{2}\right)$$
$$= \sin 3A = 3\sin A - 4\sin^3 A$$
$$= 3\frac{a}{c} - 4\frac{a^3}{c^3} = \frac{a}{c^3}(3c^2 - 4a^2).$$

7. $$\text{Area} = \frac{1}{2}ab = \frac{1}{2}ab = \frac{1}{4}\{(a+b)^2 - (a^2+b^2)\}$$
$$= \frac{1}{4}\{(a+b)^2 - c^2\} = \frac{(a+b+c)(a+b-c)}{4}.$$

8. The hypothenuse must be the diameter of the circumscribing circle;

$$\therefore R = \frac{c}{2}; \text{ and } r = \frac{S}{s} = \frac{ab}{2s} = \frac{ab}{a+b+c};$$

hence $$R + r = \frac{c}{2} + \frac{ab}{a+b+c} = \frac{(a+b)c + c^2 + 2ab}{2(a+b+c)}$$
$$= \frac{(a+b)c + a^2 + b^2 + 2ab}{2(a+b+c)}$$
$$= \frac{(a+b)(a+b+c)}{2(a+b+c)} = \frac{1}{2}(a+b).$$

Ex. 14.

3. $$\frac{\tan B}{\tan C} = \frac{\sin B}{\sin C} \times \frac{\cos C}{\cos B} = \frac{b}{c} \times \frac{a^2+b^2-c^2}{2ab} \div \frac{a^2+c^2-b^2}{2ac}$$
$$= \frac{a^2+b^2-c^2}{a^2+c^2-b^2}.$$

5. $$\frac{a^2}{c^2} - \frac{b^2}{c^2} = \frac{\sin^2 A}{\sin^2 C} - \frac{\sin^2 B}{\sin^2 C};$$

$$\therefore \frac{a^2 - b^2}{c^2} = \frac{\sin^2 A - \sin^2 B}{\sin^2 C} = \frac{\sin(A+B)\sin(A-B)}{\sin^2(A+B)} \text{ by Ex. 2, 30,}$$

$$= \frac{\sin(A-B)}{\sin(A+B)}.$$

9. $$\tan\frac{A}{2}\tan\frac{B}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$= \frac{s-c}{s} = \frac{a+b-c}{a+b+c}.$$

14. $$\text{Area} = \frac{1}{2}bc\sin A = \frac{1}{2}(a^2 - b^2)\frac{bc}{a^2 - b^2}\sin A$$

$$= \frac{1}{2}(a^2 - b^2)\frac{\sin B \sin C}{\sin^2 A - \sin^2 B}\sin A$$

$$= \frac{1}{2}(a^2 - b^2)\frac{\sin B \sin C}{\sin(A+B)\sin(A-B)}\sin A$$

$$= \frac{1}{2}(a^2 - b^2)\frac{\sin A \sin B}{\sin(A-B)}, \text{ since } \sin C = \sin(A+B).$$

15. $A + B + C = 180°$; $\therefore \frac{1}{2}(A + B - C) = 90° - C$.

Hence, $$\frac{a^2 + b^2 - c^2}{4\tan\frac{1}{2}(A+B-C)} = \frac{2ab\cos C}{4\cot C} = \frac{1}{2}ab\sin C = \text{area}.$$

Also, $$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \frac{abc}{s}\sqrt{\frac{s(s-a)}{bc}\cdot\frac{s(s-b)}{ac}\cdot\frac{s(s-c)}{ab}}$$

$$= \frac{2abc}{a+b+c}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

17. $$\frac{1}{R} = \frac{2\sin A}{a} = 2\left(\frac{\sin^3 A}{a^3}\right)^{\frac{1}{3}} = 2\left\{\frac{\sin A}{a}\cdot\frac{\sin B}{b}\cdot\frac{\sin C}{c}\right\}^{\frac{1}{3}}$$

$$= 2\left(\frac{\sin A \sin B \sin C}{abc}\right)^{\frac{1}{3}}.$$

19. $a\cos A + b\cos B + c\cos C$

$$= \frac{a}{2\sin A}\sin 2A + \frac{b}{2\sin B}\sin 2B + \frac{c}{2\sin C}\sin 2C$$
$$= R(\sin 2A + \sin 2B + \sin 2C)$$
$$= R\{2\sin(A+B)\cos(A-B) + 2\sin C\cos C\}$$
$$= 2R\sin C\{\cos(A-B) - \cos(A+B)\}$$
$$= 2R\sin C \,.\, 2\sin A\sin B, \text{ since } \cos C = -\cos(A+B)$$
$$= 4R\sin A\sin B\sin C.$$

20. Let O be the centre of the circle inscribed in the triangle ABC,

$$\text{then } \angle AOB = \gamma = \pi - \left(\frac{A}{2} + \frac{B}{2}\right) = \frac{\pi}{2} + \frac{C}{2}.$$

$$\text{Similarly, } \alpha = \frac{\pi}{2} + \frac{A}{2}; \text{ and } \beta = \frac{\pi}{2} + \frac{B}{2};$$

$$\therefore 4\sin\alpha\sin\beta\sin\gamma = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$
$$= \sin A + \sin B + \sin C, \text{ by Ex. } 6_4.$$

Ex. 15.

5. Now $\angle AEC = B + \frac{C}{2}$, and $\pi - AEC = A + \frac{C}{2}$;

$$\therefore \pi - 2AC = A - B;$$

$$\therefore \tan AEC = \cot\frac{1}{2}(A-B) = \frac{a+b}{a-b}\tan\frac{1}{2}C.$$

$$\text{Again, } AE = CE\frac{\sin\frac{1}{2}C}{\sin A}, \text{ and } BE = CE\frac{\sin\frac{1}{2}C}{\sin B};$$

$$\therefore C = AE + BE = CE\sin\frac{1}{2}C\left(\frac{1}{\sin A} + \frac{1}{\sin B}\right),$$

$$2c\,.\cos\frac{1}{2}C = CE\left(\frac{\sin C}{\sin A} + \frac{\sin C}{\sin B}\right) = CE\left(\frac{c}{a} + \frac{c}{b}\right);$$

$$\therefore CE = \frac{2ab}{a+b}\cos\frac{1}{2}C.$$

7. $\tan\frac{1}{2}B\tan\frac{1}{2}C=\left\{\frac{(p-a)(p-c)}{p(p-b)}\right\}^{\frac{1}{2}}\times\left\{\frac{(p-a)(p-b)}{p(p-c)}\right\}^{\frac{1}{2}}=\frac{p-a}{p}$;

$\therefore a=p(1-\tan\frac{1}{2}B\tan\frac{1}{2}C)=p\cos\frac{1}{2}(B+C)\div\cos\frac{1}{2}B\cos\frac{1}{2}C$

$$=p\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}B\cos\frac{1}{2}C}.$$

Similarly, $b=p\dfrac{\sin\frac{1}{2}B}{\cos\frac{1}{2}A\cos\frac{1}{2}C}$; $c=p\dfrac{\sin\frac{1}{2}C}{\cos\frac{1}{2}A\cos\frac{1}{2}B}$.

8. Let the angles be $2\theta, 3\theta, 4\theta$; then $9\theta=180°$, and the angles are $40°, 60°, 80°$,

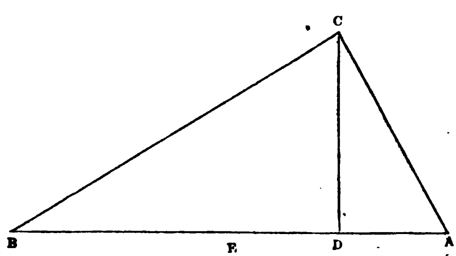
$$A=40°,\quad B=60°=\frac{3A}{2},\quad C=80°=2A.$$

Now $\dfrac{a+c}{b}=\dfrac{\sin A+\sin C}{\sin B}=\dfrac{\sin A+\sin 2A}{\sin\frac{3A}{2}}=\dfrac{2\sin\frac{3A}{2}\cos\frac{A}{2}}{\sin\frac{3A}{2}}$;

$$\therefore \cos\frac{A}{2}=\frac{a+c}{2b}.$$

9. The solution may be deduced from that of 7.

12. Bisect AB in E; then we have



$ED=AE-AD=\frac{c}{2}-b\cos A$

$$=\frac{c}{2}-c\frac{\sin B}{\sin C}\cos A=\frac{c}{2}\left\{1-\frac{2\sin B\cos A}{\sin(A+B)}\right\}$$

$$=\frac{c}{2}\left\{\frac{\sin A\cos B-\cos A\sin B}{\sin A\cos B+\cos A\sin B}\right\}=\frac{c}{2}\cdot\frac{\tan A-\tan B}{\tan A+\tan B}.$$

13. $n=\dfrac{b-a}{c}=\dfrac{\sin B-\sin A}{\sin C}=\dfrac{\sin(A+C)-\sin A}{\sin C}$

$$=\frac{2\sin\frac{C}{2}\cos\left(A+\frac{C}{2}\right)}{2\sin\frac{C}{2}\cos\frac{C}{2}};$$

$$\therefore\ \cos\left(A+\frac{C}{2}\right)=n\cos\frac{C}{2}.$$

$$\cot\frac{B-A}{2}=\frac{\cos\frac{B-A}{2}}{\sin\frac{B-A}{2}}\cdot\frac{\cos\frac{B+A}{2}}{\cos\frac{B+A}{2}}=\frac{\cos B+\cos A}{\sin B-\sin A}$$

$$=\frac{\cos B+\cos A}{n\sin C}=\frac{\sin B\cos B+\sin B\cos A}{n\sin B\sin C}$$

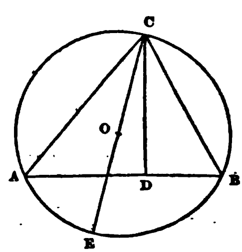
$$=\frac{\sin C-\cos B\sin A+\sin B\cos B}{n\sin B\sin C},$$

since $\sin C=\sin(A+B)$

$$=\frac{1+\frac{\sin B-\sin A}{\sin C}\cos B}{n\sin B}$$

$$=\frac{1+n\cos B}{n\sin B}.$$

20. Let ABC be any triangle; $ACBE$ the circumscribing circle, of which CE is a diameter. Draw CD perpendicular to AB.



Then $CE \,.\, CD=AC \,.\, BC$;

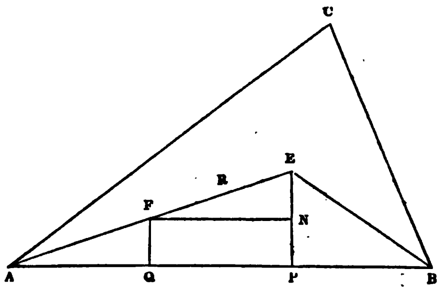
$\therefore\ CE \,.\, CB\sin B=AC \,.\, CB$;

$\therefore\ b=2R\sin B.$

Similarly, $a=2R\sin A$,

and $c=2R\sin C$.

24. Let $EP = ER = r$, $FQ = FR = r_1$;



Then $r_1 = FQ = EP - EN = r - (r + r_1)\sin\frac{A}{2}$;

$$\therefore r_1\left(1 + \sin\frac{A}{2}\right) = r\left(1 - \sin\frac{A}{2}\right),$$

$$\frac{r_1}{r} = \frac{1 - \sin\frac{A}{2}}{1 + \sin\frac{A}{2}} = \frac{\left(\cos\frac{A}{4} - \sin\frac{A}{4}\right)^2}{\left(\cos\frac{A}{4} + \sin\frac{A}{4}\right)^2} = \tan^2\left(45^\circ - \frac{A}{4}\right).$$

Similarly, $\frac{r_2}{r} = \tan^2\left(45^\circ - \frac{B}{4}\right)$, and $\frac{r_3}{r} = \tan^2\left(45^\circ - \frac{C}{4}\right)$.

Hence $(r_1r_2)^{\frac{1}{2}} + (r_1r_3)^{\frac{1}{2}} + (r_2r_3)^{\frac{1}{2}}$

$$= r\left\{\tan\frac{1}{4}(\pi - A)\tan\frac{1}{4}(\pi - B) + \tan\frac{1}{4}(\pi - A)\tan\frac{1}{4}(\pi - C)\right.$$

$$\left. + \tan\frac{1}{4}(\pi - B)\tan\frac{1}{4}(\pi - C)\right\}$$

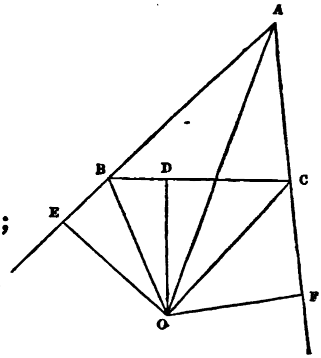
$= r$; since $\frac{1}{4}(\pi - A) + \frac{1}{4}(\pi - B) + \frac{1}{4}(\pi - C) = \frac{1}{2}\pi$.

(See Ex. 5_1)

25. If $OD = OE = OF = r_1$; then

$\Delta ABC = \text{fig. } ABOC - \Delta BOC$

$$= \frac{1}{2}AB \times OE + \frac{1}{2}AC \times OF - \frac{1}{2}BC \times OD;$$



$$\therefore\ S = \frac{1}{2} r_1 (b + c - a)$$

$$= r_1 (p - a).$$

Hence $\therefore\ S = r_0 p = r_1 (p - a)$

$$= r_2 (p - b) = r_3 (p - c);$$

if S be the area of the triangle, and $2p$ its perimeter:

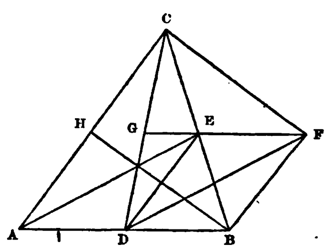
$$\therefore\ \frac{r_0}{r_1} + \frac{r_0}{r_2} + \frac{r_0}{r_3} = \frac{p-a}{p} + \frac{p-b}{p} + \frac{p-c}{p}$$

$$= \frac{3p - (a+b+c)}{p} = \frac{3p - 2p}{p} = 1,$$

$$\text{hence } \frac{1}{r_0} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

$$\text{Again, } r_0 r_1 r_2 r_3 = \frac{S}{p} \times \frac{S}{p-a} \times \frac{S}{p-b} \times \frac{S}{p-c} = S^2;$$

$$\therefore\ S = \text{area of triangle} = (r_0 r_1 r_2 r_3)^{\frac{1}{2}}.$$



26. Through D and E the middle points of AB, BC, draw DF, GEF respectively parallel to AE, AB, and join CF, BF, DE; then, because AF is a parallelogram, $DF = AE$, and EF is parallel and equal to AD or DB; and therefore BF is parallel and equal to DE or HC, and consequently FC is equal to HB. Hence DFC is the required triangle, and its area

$$= \Delta DCB \times \frac{GF}{DB} = \Delta DCB \times \left(1 + \frac{GE}{DB}\right)$$

$$= \frac{3}{2} \Delta DCB = \frac{3}{4} \Delta ACB,$$

or $\Delta ACB : \Delta DFC = 4 : 3$. (Hymers' *Trig.*)

27. Now $\dfrac{AG}{GB}=\dfrac{AC}{CB}=\dfrac{b}{a}$;

$$\therefore \frac{AG}{AB}=\frac{b}{a+b}; \quad \therefore AG=\frac{bc}{a+b}.$$

Also $\dfrac{AF}{FC}=\dfrac{AB}{BC}=\dfrac{c}{a}$; $\quad\therefore AF=\dfrac{bc}{a+c}$;

$$\therefore \Delta AFG=\frac{1}{2}\frac{b^2c^2\sin A}{(a+b)(a+c)}=\Delta ABC\times\frac{bc}{(a+b)(a+c)}.$$

Similarly, $\Delta BGE=\Delta ABC\times\dfrac{ac}{(a+b)(b+c)}$,

and $\Delta CEF=\Delta ABC\times\dfrac{ab}{(a+c)(b+c)}$;

$$\therefore \Delta EFG=\Delta ABC\left\{1-\frac{bc}{(a+b)(a+c)}-\frac{ac}{(a+b)(b+c)}-\frac{ab}{(a+c)(b+c)}\right\}$$

$$=\Delta ABC\times\frac{2abc}{(a+b)(a+c)(b+c)}.$$

28. Let $2a$ be a side of the equilateral triangle. Then

$$\frac{1}{2}(4a^2)\sin 60^\circ \text{ or } a^2\sqrt{3}=\text{its area.}$$

If $2a-d$, $2a$, $2a+d$ be the sides of the other triangle, its area

$$=\{3a\,.\,a(a+d)(a-d)\}^{\frac{1}{2}}=a\sqrt{3}\,(a^2-d^2)^{\frac{1}{2}}.$$

Hence $a\sqrt{3}\,(a^2-d^2)^{\frac{1}{2}} : a^2\sqrt{3}=3:5$;

$$\therefore a^2-d^2=\frac{9}{25}a^2; \quad \therefore d=\frac{4}{5}a.$$

The ratios of the sides are

$$2a-\frac{4}{5}a : 2a : 2a+\frac{4}{5}a=3:5:7.$$

If θ be the greatest angle, $2a+d$ is the side opposite;

$$\therefore \cos\theta=\frac{3^2+5^2-7^2}{2\times3\times5}=-\frac{1}{2}=\cos 120^\circ.$$

33. Let A and B be the areas of the inscribed and circumscribed polygons of the same number of sides; and A' the area of the inscribed polygon of twice the number of sides; then

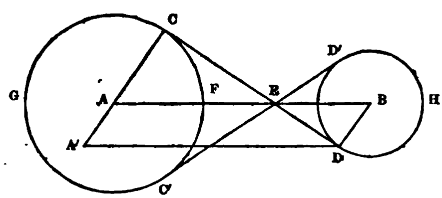
$$A = nr^2 \sin\frac{\pi}{n} \cos\frac{\pi}{n}, \quad B = nr^2 \sin\frac{\pi}{n} \div \cos\frac{\pi}{n},$$

$$\text{also} \quad A' = 2nr^2 \sin\frac{\pi}{2n} \cos\frac{\pi}{2n} = nr^2 \sin\frac{\pi}{n},$$

$$\text{hence} \quad A \times B = n^2 r^4 \sin^2\frac{\pi}{n} = A'^2,$$

$$\text{or} \quad A' = \sqrt{A \times B}.$$

35. Produce CA to A', making $AA' = BD$, join $A'D$. The length of string $= CD + C'D' + \text{arc } CGC' + \text{arc } DHD'$.



$$\text{Now} \quad CD = (A'D^2 - CA'^2)^{\frac{1}{2}} = (a^2 - c^2)^{\frac{1}{2}},$$

$$\text{arc } CF = AC \cos^{-1}\frac{AC}{AE} = AC \cos^{-1}\frac{CA'}{A'D} = AC \cos^{-1}\frac{c}{a};$$

$$\therefore \text{arc } CGC' = 2AC\left(\pi - \cos^{-1}\frac{c}{a}\right).$$

$$\text{Similarly,} \quad \text{arc } DHD' = 2BD\left(\pi - \cos^{-1}\frac{c}{a}\right);$$

$$\therefore \text{length of string} = 2\left\{(a^2 - c^2)^{\frac{1}{2}} + c\left(\pi - \cos^{-1}\frac{c}{a}\right)\right\}.$$

TRIGONOMETRICAL TABLES.

Ex. 16.

1. $\sin 123° 14' 20'' = \sin (180° - 123° 14' 20'')$

$$= \sin 56° 45' 40'',$$

$$\begin{aligned} \text{and } \sin 56° 46' &= \cdot 836446, \\ \sin 56° 45' &= \underline{\cdot 836286}, \\ \text{Diff. for } 1' \text{ or } 60'' &= + \cdot 000160; \end{aligned}$$

$$\therefore \text{ diff. for } 40'' = \frac{40}{60} \times \cdot 000160 = \cdot 000107;$$

$$\therefore \sin 123° 14' 20'' = \cdot 836286 + \cdot 000107 = \cdot 836393.$$

2. Here $\cos 41° 14' = \cdot 752032,$

$$\begin{aligned} \cos 41° 13' &= \underline{\cdot 752223}, \\ \text{Diff. for } 1' \text{ or } 60'' &= - \cdot 000191; \end{aligned}$$

$$\therefore \text{ diff. for } 26'' = \frac{26}{60} \times - \cdot 000191 = - \cdot 000083;$$

$$\therefore \cos 41° 13' 26'' = \cdot 752223 - \cdot 000083 = \cdot 752140.$$

4. $\sin A = \cdot 346105$ $\qquad \sin 20° 15' = \cdot 346117$

$\sin 20° 14' = \underline{\cdot 345844}$ $\qquad \sin 20° 14' = \underline{\cdot 345844}$

261 $\qquad 273$

Hence $273 : 261 :: 60'' : (n)$ no. of seconds required;

$$\therefore n = 60 \times \frac{261}{273} = 57'' \ldots;$$

$$\therefore A = 20° 14' 57''.$$

7. $\log \tan 23° 24' = 9 \cdot 636226$, $\quad 100'' : 25'' :: 577 :$ req. correction

corr. for $25'' = \underline{144}$ $\qquad \underline{\cdot 25}$

$\log \tan 23° 24' 25'' = 9 \cdot 636370$ $\qquad 144 \cdot 25$

9. $\log \cos A = 9 \cdot 123456$

$\log \cos 82° \, 21' = \underline{9 \cdot 124248}$

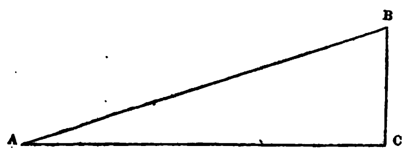
$\therefore 1566 : 792 :: 100'' :$ required number of seconds $= 50''$;

$\therefore A = 82° \, 21' \, 50''$.

Solution of Plane Triangles.

Ex. 17.

1. $A + B = 90°$; $\therefore B = 90° - 18° \, 14' = 71° \, 46'$;



$$\therefore \frac{BC}{AB} = \sin A;$$

$\therefore \log BC = \log AB + \log \sin A - 10$

$= 2 \cdot 130872$ $\qquad$ $\log 432 = 2 \cdot 635484$

$\log 135 \cdot 1 = \underline{2 \cdot 130655}$ $\qquad$ $\log \sin 18° \, 14' = \underline{9 \cdot 495388}$

217 $\qquad$ $12 \cdot 130872$

$6 \quad \underline{192}$

250

$7 \quad \underline{224}$

$\therefore BC = 135 \cdot 167$;

$\therefore AC = \sqrt{432^2 - (135 \cdot 167)^2} = \sqrt{567 \cdot 167 \times 296 \cdot 833} = 410 \cdot 31$ by logs.

8. $$C = 180° - (A + B) = 100° \; 22' \; 45'',$$

$$AB = BC . \frac{\sin C}{\sin A}; \quad \therefore \log AB = \log BC + \log \sin C - \log \sin A,$$

$$AC = BC . \frac{\sin B}{\sin A}; \quad \therefore \log AC = \log BC + \log \sin B - \log \sin A,$$

$\log BC \; (= 9267\cdot)$	$= 2\cdot966939$	$\log BC$	$= 2\cdot966939$
$\log \sin C \; (= 100° \; 22' \; 43'')$	$= 9\cdot992835$	$\log \sin B$	$= 9\cdot873812$
*A.C. $\log \sin A \; (= 31° \; 13')$	$= 0\cdot285439$	A.C. $\log \sin A$	$= 0\cdot285439$
$\therefore \log AB$	$= 3\cdot245213$	$\therefore \log AC$	$= 3\cdot126190$
$\therefore AB$	$= 1758\cdot78,$	AC	$= 1337\cdot18.$

15. $$\tan \frac{1}{2} (C - B) = \frac{AB - AC}{AB + AC} \tan \frac{1}{2} (C + B) = \frac{72}{780} \tan 65° \; 22';$$

$$\therefore \log \tan \frac{1}{2} (C - B) = \log 72 + \log \tan 65° \; 22' - \log 780$$

$$= 9\cdot303861$$

$$= \log \tan 11° \; 22' \; 55''$$

$\log 72 =$	$1\cdot857333$
A.C. $\log 780 =$	$7\cdot107905$
$\log \tan 65° \; 22' =$	$10\cdot338623$
	-10
	$9\cdot303861$

Hence, $\frac{1}{2} (C + B) = 65° \; 22',$

$$\frac{1}{2} (C - B) = 11° \; 22' \; 55'';$$

$$\therefore \; C = 76° \; 44' \; 55'', \text{ and } B = 53° \; 59' \; 5''.$$

Again, $$BC = AC \frac{\sin A}{\sin B},$$

$$\log BC = \log AC + \log \sin A - \log \sin B$$

$$= \log 331\cdot63,$$

$$\therefore BC = 331\cdot63.$$

$\log 354 =$	$2\cdot549003$
$\log \sin 49° \; 16' =$	$9\cdot879529$
A.C. $\log \sin 53° \; 59' \; 5'' =$	$0\cdot092127$
	-10
	$2\cdot520659$
$\log 331\cdot6 =$	$2\cdot520614$
pp. 3	45
	39

* Note. A.C. denotes Arithmetic Complement.

17. Here $s = \dfrac{1263 + 1359 + 1468}{2} = \dfrac{4090}{2} = 2045$;

$$\therefore s - a = 2045 - 1468 = 577;$$

$$\therefore \cos \frac{1}{2} A = \left\{\frac{s(s-a)}{bc}\right\}^{\frac{1}{2}} = \left\{\frac{2045 \times 577}{1359 \times 1263}\right\}^{\frac{1}{2}};$$

$$\therefore \log \cos \frac{1}{2} A = \frac{1}{2}\{\log 2045 + \log 577 - \log 1359 - \log 1263\},$$

$$\begin{aligned} \log 2045 &= 3{\cdot}310693 \\ \log 577 &= 2{\cdot}761176 \\ \text{A.C.} \log 1359 &= \bar{4}{\cdot}866781 \\ \text{A.C.} \log 1263 &= \bar{4}{\cdot}898596 \\ 2\,)\,&\underline{9{\cdot}837246} \end{aligned}$$

$$\therefore \log \cos \frac{1}{2} A = 9{\cdot}918623 = \log \cos 33^\circ\, 59'\, 25''\tfrac{1}{2} \text{ nearly};$$

$$\therefore \frac{A}{2} = 33^\circ\, 59'\, 25''\tfrac{1}{2}, \quad A = 67^\circ\, 58'\, 51''.$$

Now $\sin C = \sin A \cdot \dfrac{AB}{BC} = \dfrac{1263}{1468} \sin 67^\circ\, 58'\, 51''$,

whence $C = 52^\circ\, 54'\, 5''$; $\therefore B = 180^\circ - (A + C) = 59^\circ\, 7'\, 4''$.

19. By question, $c : b :: 7 : 4$;

$$\therefore c - b : c + b :: 3 : 11;$$

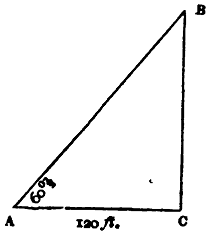
$$\therefore \tan \frac{1}{2}(C - B) = \frac{c-b}{c+b} \cot \frac{A}{2} = \frac{3}{11} \cot 4^\circ\, 37'\, 47{\cdot}5'',$$

$$\begin{aligned} \log \cot 4^\circ\, 37'\, 47{\cdot}5'' &= 11{\cdot}091609 \\ \log 3 &= 0{\cdot}477121 \\ \text{A.C.} \log 11 &= \underline{\bar{2}{\cdot}958607} \\ \therefore \log \tan \frac{1}{2}(C - B) &= 10{\cdot}527337 \end{aligned}$$

$$\begin{aligned} \text{whence } \frac{C - B}{2} &= 73^\circ\, 27'\, 43'', \\ \text{but } \frac{C + B}{2} &= \underline{85^\circ\, 22'\, 12''\tfrac{1}{2}} \\ \therefore C &= 158^\circ\, 49'\, 55''\tfrac{1}{2} \\ B &= 11^\circ\, 54'\, 29''\tfrac{1}{2} \end{aligned}$$

Heights and Distances.

Ex. 18.



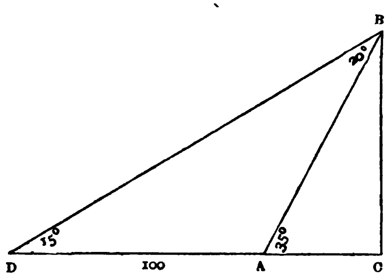
1. $BC = AC \,.\, \tan BAC$;

$\therefore \log BC = \log 120 + \log \tan 60° \, 30' - 10$

$= 2{\cdot}326539 = \log 212{\cdot}099$;

$\therefore$ BC the height $= 212{\cdot}099$ feet.

3. $$AB = \frac{AD \,.\, \sin 15°}{\sin 20°};$$



$\therefore \log AB = \log AD + \log \sin 15° - \log \sin 20°$
$= 1{\cdot}878944 = \log 75{\cdot}6735$;

$\therefore$ length of ladder $= 75{\cdot}6735$ yd.

$BC = AB \sin A$;

$\therefore \log BC = \log AB + \log \sin A - 10$
$= 1{\cdot}637535 = \log 43{\cdot}4045$;

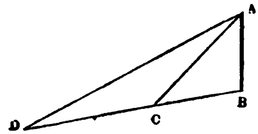
$\therefore$ height of wall $= 43{\cdot}4045$ yd.

$AC = AB \cos 35°$;

$\therefore \log AC = \log AB + \log \cos 35° - 10$
$= 1{\cdot}792309 = \log 61{\cdot}988$;

$\therefore$ breadth of ditch $= 61{\cdot}988$ yd.

17. Given $BC = 40$, $CD = 60$,



$$ACB = 34^\circ\ 18'\ 19'',\quad ADB = 19^\circ\ 14'\ 52'',$$

$$\text{and } \therefore\ DAC = 15^\circ\ 3'\ 27''.$$

$$AC = CD\,\frac{\sin ADC}{\sin DAC};$$

$$\therefore\ \log AC = \log CD + \log\sin ADC - \log\sin DAC$$

$$= 1{\cdot}881589 = \log 76{\cdot}1358;$$

$$\therefore\ b + a = 116{\cdot}1358;\quad b - a = 36{\cdot}1358;$$

$$\therefore\ \tan\frac{1}{2}(B - A) = \frac{b-a}{b+a}\cot\frac{1}{2}C = \frac{36{\cdot}1358}{116{\cdot}1358}\cot 17^\circ\ 9'\ 9{\cdot}5'',$$

$$\begin{aligned}
\log 36{\cdot}1358 &= 1{\cdot}557938\\
\text{A.C. } \log 116{\cdot}1358 &= \bar{3}{\cdot}935034\\
\log\cot 17^\circ\ 9'\ 9{\cdot}5'' &= \underline{10{\cdot}510539}
\end{aligned}$$

$$\therefore\ \log\tan\frac{B-A}{2} = 10{\cdot}003511 = \log\tan 45^\circ\ 13'\ 53{\cdot}5'';$$

$$\therefore\ \frac{1}{2}(B - A) = 45^\circ\ 13'\ 53{\cdot}5''$$

$$\frac{1}{2}(B + A) = 72^\circ\ 50'\ 50{\cdot}5''$$

$$\therefore\ B = 117^\circ\ 4'\ 44''$$

$$A = 27^\circ\ 36'\ 57''$$

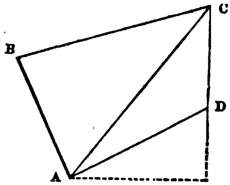
$$\text{Again } AB = BC.\frac{\sin ACB}{\sin CAB},$$

$$\begin{aligned}
\text{and } \log BC\ (=40) &= 1{\cdot}602060\\
\log\sin ACB\ (34^\circ\ 18'\ 19'') &= 9{\cdot}750963\\
\text{A.C. } \log\sin CAB\ (27^\circ\ 36'\ 57'') &= \underline{0{\cdot}333912}
\end{aligned}$$

$$\therefore\ \log AB = 1{\cdot}686935 = \log 48{\cdot}633;$$

$$\therefore\ AB = 48{\cdot}633 \text{ ft.}$$

18. Given $AB = 100$,



$$\left.\begin{array}{l}\angle BAC = 73^\circ\ 14' \\ \angle ABC = 73^\circ\ 18'\end{array}\right\}; \therefore \angle BCA = 33^\circ\ 28',$$

$$\left.\begin{array}{l}\angle CAD = 34^\circ\ 30' \\ \angle ACD = 44^\circ\ 45'\end{array}\right\}; \therefore \angle ADC = 100^\circ 45',$$

$$AC = AB\frac{\sin ABC}{\sin ACB},$$

$$\log AC = \log AB + \log\sin ABC - \log\sin ACB$$
$$= 2{\cdot}239777.$$

$$\text{Again,}\quad CD = AC\frac{\sin CAD}{\sin ADC},$$

$$\log CD = \log AC + \log\sin CAD - \log\sin ADC$$
$$= 2{\cdot}000594 = \log 100{\cdot}137;$$

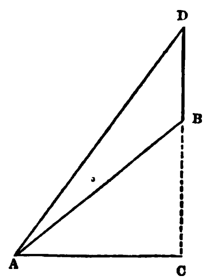
$\therefore CD = 100{\cdot}137$ yd. = the height of the castle.

20. Let $\angle DAB = 1^\circ\ 54'\ 10'' = \alpha$,

$AC = 250$,

$BD = 12$,

$CAB = \theta$.



Then $\tan(\theta + \alpha) - \tan\theta$

$$= \frac{DC}{AC} - \frac{BC}{AC} = \frac{BD}{AC} = \frac{12}{250};$$

$$\therefore \frac{\sin(\theta + \alpha - \theta)}{\cos(\theta + \alpha)\cos\theta} = \frac{12}{250},$$

$$2\cos(\theta + \alpha)\cos\theta \text{ or } \cos(2\theta + \alpha) + \cos\alpha = \frac{500\sin\alpha}{12} = 1{\cdot}3835,$$

$$\cos(2\theta + \alpha) = 1{\cdot}3835 - {\cdot}999448 = {\cdot}38405 = \cos 67^\circ\ 24'\ 55'';$$

$$\therefore \theta = \frac{1}{2}(67^\circ\ 24'\ 55'' - 1^\circ\ 54'\ 10'') = 32^\circ\ 45'\ 22{\cdot}5''.$$

Now $BC = AC \tan\theta$,

$= 160{\cdot}85$ nearly.

$$\begin{aligned} \log 250 &= 2{\cdot}397940 \\ \log\tan 32^\circ\, 45''\, 22{\cdot}5' &= 9{\cdot}808468 \\ &\quad -10 \\ \hline &\quad 2{\cdot}206408 \\ \log 160{\cdot}8 &= 2{\cdot}206286 \\ \hline &\quad 122 \end{aligned}$$

23. Let $AB - AC = \delta$,

$AB - BD = \delta'$,

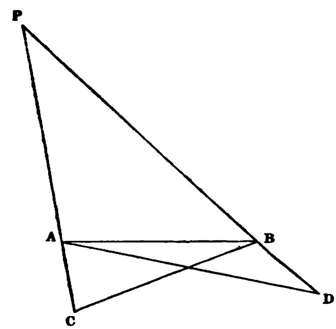
$BC = a$,

$AD = a'$,

$AC = b$,

$BD = b'$,

$AB = c$.



Then,

$$\sin\frac{BAC}{2} = \sqrt{\frac{(a+\delta)(a-\delta)}{bc}}; \quad \sin\frac{ABD}{2} = \sqrt{\frac{(a'+\delta')(a'-\delta')}{b'c}}.$$

Hence, we find $\angle BAC = 79^\circ\, 55'\, 17''$, $\angle ABD = 144^\circ\, 9'\, 57''$;

$$\therefore BAP = 100^\circ\, 4'\, 43', \quad ABP = 35^\circ\, 50'\, 3';$$

$$\text{and } \therefore APB = 44^\circ\, 5'\, 14''.$$

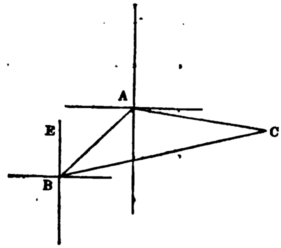
Now $PA = AB\dfrac{\sin ABP}{\sin APB}$;

$$\begin{aligned} \log AB &= 2{\cdot}698970 \\ \log\sin ABP &= 9{\cdot}767483 \\ \text{A.C.} \log\sin APB &= 0{\cdot}157545 \\ \hline \therefore \log AP &= 2{\cdot}623998 \\ \therefore AP &= 420{\cdot}72 \text{ yd.} \end{aligned}$$

and $PB = AB\dfrac{\sin BAP}{\sin APB}$,

$$\begin{aligned} \log AB &= 2{\cdot}698970 \\ \log\sin BAP &= 9{\cdot}993246 \\ \text{A.C.} \log\sin APB &= 0{\cdot}157545 \\ \hline \therefore \log BP &= 2{\cdot}849761 \\ BP &= 707{\cdot}56 \text{ yd.} \end{aligned}$$

26. Let A be the harbour; B the position of the privateer, and AC the direction in which the merchantman is sailing.



By question $AB = 10$,

$$AC = 13\cdot5,$$

$$\angle BAC = 125^\circ;$$

$$\therefore \tan\frac{1}{2}(B - C) = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{3\cdot5}{23\cdot5}\cot 62^\circ\, 30',$$

when $\frac{1}{2}(B - C) = 4^\circ\, 26'$, and $\frac{1}{2}(B + C) = 27^\circ\, 30'$, by question,

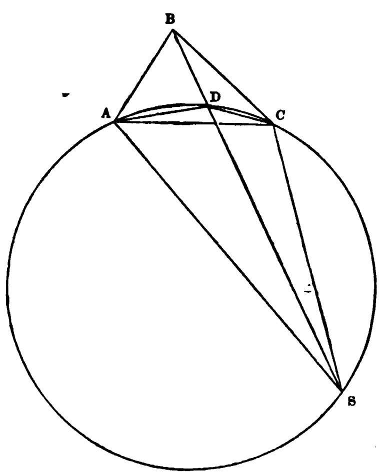
$$\therefore B = 31^\circ\, 56', \quad C = 23^\circ\, 4';$$

$\therefore \angle EBC = 45^\circ + 31^\circ\, 56' =$ N $76^\circ\, 56'$ E, the direction in which the privateer sails.

Again, $BC = AB\dfrac{\sin BAC}{\sin ACB}$, whence $BC = 20\cdot907$ miles;

$$\therefore 20\cdot907 \div 1\tfrac{1}{2} = 13\cdot938 \text{ miles per hour.}$$

31. Let A, B, C denote the positions of the three objects.



At A draw a line, making an

$$\angle CAD = \angle CSB = 8^\circ\, 53'.$$

At C draw a line, making an

$$\angle ACD = \angle ASB = 14^\circ\, 16'.$$

Of these two lines, let the intersection be D, and about the $\Delta\, ADC$ describe a circle; join BD and produce it to meet the circumference in S; join AS, CS.

(1) Find AD, from the data, AC, $\angle$s CAD, ACD.

(2) ...... $\angle BAC$, ... AB, AC, BC.

(3) ...... $\angle$sABD, ADB ...... AB, AD, $\angle BAD$.

(4) ...... AS, BS .... AB, $\angle$s ABS, ASB.

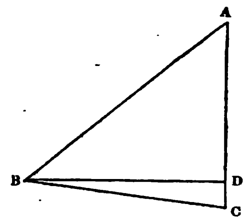
(5) ...... CS .... AC, $\angle$s ACS, ASC.

36. Let $\angle ABD = \theta$,

$\angle DBC = \phi$,

$AC = 216$,

$CD = 5$.



Then $\dfrac{\tan\theta}{\tan\phi} = \dfrac{AD}{DC} = \dfrac{211}{5}$;

$$\therefore \frac{\tan\theta - \tan\phi}{\tan\theta + \tan\phi} = \frac{206}{216} = \frac{103}{108};$$

$$\therefore \sin(\theta - \phi) = \frac{103}{108}\sin(\theta + \phi) = \frac{103}{108}\sin 47^\circ\ 56'.$$

Hence $\theta = \frac{1}{2}\{(\theta + \phi) + (\theta - \phi)\}$ is easily determined,

and $BD = AD\cot\theta$, gives the required distance $= 200{\cdot}21$ ft.

Expansions, Series, &c.

Ex. 19.

1. Let $2\cos\theta = x + x^{-1}$; then $2\cos n\theta = x^n + x^{-n}$;

$$\begin{aligned}\therefore (2\cos\theta)^6 &= x^6 + 6x^4 + 15x^2 + 20 + 15x^{-2} + 6x^{-4} + x^{-6}\\ &= (x^6 + x^{-6}) + 6(x^4 + x^{-4}) + 15(x^2 + x^{-2}) + 20\\ &= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20;\end{aligned}$$

$$\therefore \cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}.$$

3. Let $2\cos\theta = x + x^{-1}$; then $2\sqrt{-1}\sin\theta = x - x^{-1}$,

and $(2\sqrt{-1}\sin\theta)^4 = (x - x^{-1})^4 = x^4 - 4x^2 + 6 - 4x^{-2} + x^{-4}$

$$= (x^4 + x^{-4}) - 4(x^2 + x^{-2}) + 6,$$

$$2^4 \sin^4\theta = 2\cos 4\theta - 8\cos 2\theta + 6;$$

$$\therefore \sin^4\theta = \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}.$$

5. Let $2\cos\theta = x + x^{-1}$; then $2\sqrt{-1}\sin\theta = x - x^{-1}$;

$$\therefore (2\sqrt{-1}\sin\theta)^5 = x^5 - 5x^3 + 10x - 10x^{-1} + 5x^{-3} - x^{-5}$$

$$= (x^5 - x^{-5}) - 5(x^3 - x^{-3}) + 10(x - x^{-1}),$$

$$2^5\sqrt{-1}\sin^5\theta = 2\sqrt{-1}\sin 5\theta - 10\sqrt{-1}\sin 3\theta + 20\sqrt{-1}\sin\theta;$$

$$\therefore \sin^5\theta = \frac{1}{16}\{\sin 5\theta - 5\sin 3\theta + 10\sin\theta\}.$$

7. By Demoivre's Theorem,

$$\cos 4\theta + \sqrt{-1}\sin 4\theta = (\cos\theta + \sqrt{-1}\sin\theta)^4$$

$$= \cos^4\theta + 4\sqrt{-1}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta$$

$$- 4\sqrt{-1}\cos\theta\sin^3\theta + \sin^4\theta.$$

Equating the impossible terms of this equation, we have

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.$$

10. $\sqrt{-1}\tan 2\theta = \dfrac{2\sqrt{-1}\sin 2\theta}{2\cos 2\theta} = \dfrac{e^{2\theta\sqrt{-1}} - e^{-2\theta\sqrt{-1}}}{e^{2\theta\sqrt{-1}} + e^{-2\theta\sqrt{-1}}}$

$$= \frac{(e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}})(e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}})}{(e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}})^2 - 2} = \frac{2\sqrt{-1}\sin\theta\, 2\cos\theta}{(2\cos\theta)^2 - 2}$$

$$= \sqrt{-1}\,.\,\frac{2\tan\theta}{2 - \sec^2\theta}; \quad \therefore \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

11. $2\cos 2\theta = e^{2\theta\sqrt{-1}} + e^{-2\theta\sqrt{-1}}$

$$= \frac{(e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}})^2 + (e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}})^2}{2}$$

$$= \frac{(2\cos\theta)^2 + (2\sqrt{-1}\sin\theta)^2}{2}$$

$$= 2\cos^2\theta - 2\sin^2\theta;$$

$$\therefore \cos 2\theta = \cos^2\theta - \sin^2\theta.$$

Ex. 20.

1. Let S = the required sum, and

$$2\cos x = \epsilon^{x\sqrt{-1}} + \epsilon^{-x\sqrt{-1}};$$

then $2S = 2 + (\epsilon^{x\sqrt{-1}} + \epsilon^{-x\sqrt{-1}}) + (\epsilon^{2x\sqrt{-1}} + \epsilon^{-2x\sqrt{-1}}) + \ldots$ to n terms

$$= 1 + \epsilon^{x\sqrt{-1}} + \epsilon^{2x\sqrt{-1}} + \ldots\ldots + \epsilon^{(n-1)x\sqrt{-1}}$$
$$+ 1 + \epsilon^{-x\sqrt{-1}} + \epsilon^{-2x\sqrt{-1}} + \ldots\ldots + \epsilon^{-(n-1)x\sqrt{-1}}$$

$$= \frac{\epsilon^{nx\sqrt{-1}} - 1}{\epsilon^{x\sqrt{-1}} - 1} + \frac{\epsilon^{-nx\sqrt{-1}} - 1}{\epsilon^{-x\sqrt{-1}} - 1}$$

$$= \frac{(\epsilon^{(n-1)x\sqrt{-1}} + \epsilon^{-(n-1)x\sqrt{-1}}) - (\epsilon^{nx\sqrt{-1}} + \epsilon^{-nx\sqrt{-1}}) - (\epsilon^{x\sqrt{-1}} + \epsilon^{-x\sqrt{-1}}) + 2}{2 - (\epsilon^{x\sqrt{-1}} + \epsilon^{-x\sqrt{-1}})};$$

$$\therefore S = \frac{\cos(n-1)x - \cos nx - \cos x + 1}{2 - 2\cos x}$$

$$= \frac{\sin\left(n - \frac{1}{2}\right)x \, . \sin\frac{x}{2} + \sin^2\frac{x}{2}}{2\sin^2\frac{x}{2}} = \frac{\sin\left(n - \frac{1}{2}\right)x + \sin\frac{x}{2}}{2\sin\frac{x}{2}}$$

$$= \frac{\sin\frac{nx}{2} \, . \cos\frac{(n-1)x}{2}}{\sin\frac{x}{2}}.$$

2. Let S = the required sum, and $2\cos\theta = z + \frac{1}{z}$, then

$$2\cos n\theta = z^n + \frac{1}{z^n},$$

$$2S = 2 + x(z + z^{-1}) + x^2(z^2 + z^{-2}) + \ldots\ldots + x^{n-1}(z^{n-1} + z^{-(n-1)})$$

$$= 1 + xz + (xz)^2 + \ldots\ldots + (xz)^{n-1}$$
$$+ 1 + xz^{-1} + x^2z^{-2} + \ldots\ldots + x^{n-1}z^{-(n-1)}$$

$$= \frac{(xz)^n - 1}{xz - 1} + \frac{x^nz^{-n} - 1}{xz^{-1} - 1}$$

$$= \frac{x^{n+1}\left(z^{n-1} + \frac{1}{z^{n-1}}\right) - x^n\left(z^n + \frac{1}{z^n}\right) - x\left(z + \frac{1}{z}\right) + 2}{x^2 - x\left(z + \frac{1}{z}\right) + 1};$$

$$\therefore\ S = \frac{x^{n+1}\cos(n-1)\theta - x^n\cos n\theta - x\cos\theta + 1}{x^2 - 2x\cos\theta + 1}.$$

3. Let $2\sqrt{-1}\sin\theta = \epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}$, S = the required sum; then

$$2\sqrt{-1}\,S = (\epsilon^{\theta\sqrt{-1}} - \epsilon^{-\theta\sqrt{-1}}) + (\epsilon^{2\theta\sqrt{-1}} - \epsilon^{-2\theta\sqrt{-1}}) + \ldots\ldots$$

N.B. Proceed as above in 1.

6. We have $\operatorname{cosec}\theta + \cot\theta = \cot\frac{\theta}{2}$,

hence $\operatorname{cosec}\theta = \cot\frac{1}{2}\theta - \cot\theta$,

$$\operatorname{cosec} 2\theta = \cot\theta - \cot 2\theta,$$

$$\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\operatorname{cosec} 2^{n-1}\theta = \cot 2^{n-2}\theta - \cot 2^{n-1}\theta;$$

and $\therefore\ S = \cot\frac{\theta}{2} - \cot 2^{n-1}\theta.$

10. Since $\tan\theta = \cot\theta - 2\cot 2\theta$;

$$\therefore \frac{1}{2}\tan\frac{\theta}{2} = \frac{1}{2}\cot\frac{\theta}{2} - \cot\theta,$$

$$\frac{1}{2^2}\tan\frac{\theta}{2^2} = \frac{1}{2^2}\cot\frac{\theta}{2^2} - \frac{1}{2}\cot\frac{\theta}{2},$$

$$\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{2^{n-1}}\tan\frac{\theta}{2^{n-1}} = \frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}} - \frac{1}{2^{n-2}}\cot\frac{\theta}{2^{n-2}};$$

$$\therefore \text{ the sum } S = \frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}} - 2\cot 2\theta.$$

11. Let $2\cos\theta = z + \frac{1}{z}$, and $S =$ the required sum.

Then

$$2S = 2 + \frac{x}{1}\left(z + \frac{1}{z}\right) + \frac{x^2}{1.2}\left(z^2 + \frac{1}{z^2}\right) + \frac{x^3}{1.2.3}\left(z^3 + \frac{1}{z^3}\right) + \ldots\ldots$$

$$= \epsilon^{xz} + \epsilon^{\frac{x}{z}} = \epsilon^{x(\cos\theta + \sqrt{-1}\sin\theta)} + \epsilon^{x(\cos\theta - \sqrt{-1}\sin\theta)};$$

$$\therefore S = \epsilon^{x\cos\theta} \times \frac{1}{2}\left(\epsilon^{x\sin\theta\sqrt{-1}} + \epsilon^{-x\sin\theta\sqrt{-1}}\right)$$

$$= \epsilon^{x\cos\theta}\cos(x\sin\theta).$$

14. Let $2\sqrt{-1}\sin\theta = z - \frac{1}{z}$, and $S =$ the required sum.

Then

$$2\sqrt{-1}\,S = x\left(z - \frac{1}{z}\right) - \frac{x^2}{2}\left(z^2 - \frac{1}{z^2}\right) + \frac{x^3}{3}\left(z^3 - \frac{1}{z^3}\right) - \ldots\ldots$$

$$= xz - \frac{1}{2}(xz)^2 + \frac{1}{3}(xz)^3 - \ldots - \left(\frac{x}{z} - \frac{1}{2}\frac{x^2}{z^2} + \frac{1}{3}\frac{x^3}{z^3} - \ldots\ldots\right)$$

$$= \log(1 + xz) - \log\left(1 + \frac{x}{z}\right) = \log\frac{1 + xz}{1 + \frac{x}{z}};$$

$$\therefore \epsilon^{2S\sqrt{-1}} = \frac{1 + xz}{1 + \frac{x}{z}}.$$

$$\text{Hence}\quad \frac{\epsilon^{2S\sqrt{-1}}-1}{\epsilon^{2S\sqrt{-1}}+1}=\frac{xz-\dfrac{x}{z}}{2+xz+\dfrac{x}{z}}=\frac{2x\sqrt{-1}\sin\theta}{2+2x\cos\theta};$$

$$\therefore\ \tan S=\frac{x\sin\theta}{1+x\cos\theta},\quad\text{and}\quad S=\tan^{-1}\left(\frac{x\sin\theta}{1+x\cos\theta}\right).$$

17.
$$\sin\theta=\theta-\frac{\theta^3}{1.2.3}+\frac{\theta^5}{1.2.3.4.5}\ldots\ldots$$
$$=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\ldots\ldots$$
$$=\theta-\frac{\theta^3}{\pi^2}\left\{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots\right\}$$
$$+\frac{\theta^5}{\pi^4}\left\{\frac{1}{1^2.2^2}+\frac{1}{1^2.3^2}+\frac{1}{2^2.3^2}\ldots\ldots\right\}$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Therefore equating the coefficients of θ^3,

$$\frac{1}{\pi^2}\left\{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots\right\}=\frac{1}{6};$$
$$\therefore\ \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots=\frac{\pi^2}{6}.$$

20.
$$e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}=n\left\{e^{(a+x)\sqrt{-1}}-e^{-(a+x)\sqrt{-1}}\right\};$$
$$\therefore\ \left(1-ne^{a\sqrt{-1}}\right)e^{x\sqrt{-1}}=\left(1-ne^{-a\sqrt{-1}}\right)e^{-x\sqrt{-1}},$$
$$\text{hence}\quad e^{2x\sqrt{-1}}=\frac{1-ne^{-a\sqrt{-1}}}{1-ne^{a\sqrt{-1}}},$$
$$\text{or}\quad 2x\sqrt{-1}=\log_e\left(1-ne^{-a\sqrt{-1}}\right)-\log_e\left(1-ne^{a\sqrt{-1}}\right)$$
$$=n\left(e^{a\sqrt{-1}}-e^{-a\sqrt{-1}}\right)+\frac{n^2}{2}\left(e^{2a\sqrt{-1}}-e^{-2a\sqrt{-1}}\right)$$
$$+\frac{n^3}{3}\left(e^{3a\sqrt{-1}}-e^{-3a\sqrt{-1}}\right),\ \&c.;$$
$$\therefore\ x=n\sin a+\frac{n^2}{2}\sin 2a+\frac{n^3}{3}\sin 3a+\&c.$$

22. If $2\cos A = x + \frac{1}{x}$, and $2\cos B = y + \frac{1}{y}$; then

$$\frac{a}{b} = \frac{\sin A}{\sin B} = \frac{x - x^{-1}}{y - y^{-1}}; \quad \therefore \log_e \frac{a}{b} = \frac{1}{2}\log_e \left(\frac{x - x^{-1}}{y - y^{-1}}\right)^2;$$

$$\text{or } \log_e \frac{a}{b} = \frac{1}{2}\log_e \frac{x^2 - 2 + x^{-2}}{y^2 - 2 + y^{-2}} = \frac{1}{2}\log_e \frac{(1 - x^2)(1 - x^{-2})}{(1 - y^2)(1 - y^{-2})}$$

$$= \frac{1}{2}\{\log_e (1 - x^2) + \log_e (1 - x^{-2}) - \log_e (1 - y^2) - \log_e (1 - y^{-2})\}$$

$$= \frac{1}{2}\{- (x^2 + \frac{1}{2}x^4 + \&c.) - (x^{-2} + \frac{1}{2}x^{-4} + \&c.)$$

$$+ (y^2 + \frac{1}{2}y^4 + \&c.) + (y^{-2} + \frac{1}{2}y^{-4} + \&c.)\}$$

$$= \frac{1}{2}[\{(y^2 + y^{-2}) - (x^2 + x^{-2})\} + \frac{1}{2}\{(y^4 + y^{-4}) - (x^4 + x^{-4})\} + \&c.]$$

$$= (\cos 2B - \cos 2A) + \frac{1}{2}(\cos 4B - \cos 4A) + \&c.$$

23. We have $c^2 = a^2 + b^2 - 2ab\cos C$; (if $2\cos C = x + x^{-1}$)

$$= a^2 + b^2 - ab(x + x^{-1})$$

$$= (a - bx)(a - bx^{-1}),$$

$$\text{and } \frac{c^2}{a^2} = \left(1 - \frac{b}{a}x\right)\left(1 - \frac{b}{a}x^{-1}\right);$$

$$\therefore 2\log_e \frac{c}{a} = \log_e \left(1 - \frac{b}{a}x\right) + \log_e \left(1 - \frac{b}{a}x^{-1}\right)$$

$$= -\frac{b}{a}(x + x^{-1}) - \frac{b^2}{2a^2}(x^2 + x^{-2}) - \&c.$$

$$\therefore 2\log_e \frac{a}{e} = \frac{b}{a}(x + x^{-1}) + \frac{b^2}{2a^2}(x^2 + x^{-2}) + \&c.;$$

$$\therefore \log_e \frac{a}{c} = \frac{b}{a}\cos C + \frac{b^2}{2a^2}\cos 2C + \frac{b^3}{3a^3}\cos 3C + \&c.$$

Ex. 21.

1. Since $\tan\theta = \dfrac{b}{a}$; $\therefore \cos\theta = \dfrac{a}{(a^2+b^2)^{\frac{1}{2}}}$;

$$\therefore a \pm b\sqrt{-1} = (a^2+b^2)^{\frac{1}{2}}\{\cos\theta \pm \sqrt{-1}\sin\theta\};$$

$$\therefore (a \pm b\sqrt{-1})^{\frac{1}{n}} = (a^2+b^2)^{\frac{1}{2n}}\{\cos\theta \pm \sqrt{-1}\sin\theta\}^{\frac{1}{n}}$$

$$= (a^2+b^2)^{\frac{1}{2n}}\left\{\cos\frac{\theta}{n} \pm \sqrt{-1}\sin\frac{\theta}{n}\right\}.$$

4. By the preceding example, we have

$$\sin\theta = 2^{n-1}\sin\frac{\theta}{n}\,.\,\sin\frac{\pi+\theta}{n}\,.\,\sin\frac{2\pi+\theta}{n}\ldots\ldots\sin\frac{(n-1)\pi+\theta}{n},$$

put $\dfrac{n\pi}{2}+\theta$ for θ;

$$\therefore \sin\left(\frac{n\pi}{2}+\theta\right) = 2^{n-1}\cos\frac{\theta}{n}\cos\frac{\pi+\theta}{n}\cos\frac{2\pi+\theta}{n}\ldots\cos\frac{(n-1)\pi+\theta}{n},$$

or $$\frac{\sin\theta}{\sin\left(\frac{n\pi}{2}+\theta\right)} = \tan\frac{\theta}{n}\tan\frac{\pi+\theta}{n}\ldots\ldots\tan\frac{(n-1)\pi+\theta}{n} = A.$$

Now if n be a multiple of 4, $\sin\left(\dfrac{n\pi}{2}+\theta\right) = \sin\theta$;

$$\therefore \tan\frac{\theta}{n}\tan\frac{\pi+\theta}{n}\ldots\ldots\tan\frac{(n-1)\pi+\theta}{n} = 1,$$

whatever (θ) be. Put

$$\theta = \frac{\pi}{4}, \text{ and } \tan\frac{\pi}{4n}\tan\frac{5\pi}{4n}\tan\frac{9\pi}{4n}\ldots\ldots\tan\frac{(4n-3)\pi}{4n} = 1,$$

when n is a multiple of 4.

$$\text{If } n = 2m, \ \sin\left(\frac{n\pi}{2}+\theta\right) = (-1)^m\sin\theta,$$

$$\text{and } \tan\frac{\pi}{4n}\,.\,\tan\frac{5\pi}{4n}\ldots\ldots\tan\frac{(4n-3)\pi}{4n} = (-1)^n.$$

If $n = 2m + 1$, and $\theta = \dfrac{\pi}{4}$,

$$\sin\left(\frac{n\pi}{2} + \theta\right) = \sin\left(m\pi + \frac{3\pi}{4}\right) = \cos m\pi \sin\frac{\pi}{4} = (-1)^m \sin\frac{\pi}{4};$$

$$\therefore \frac{\sin\theta}{\sin\left(\dfrac{n\pi}{2} + \theta\right)} = (-1)^m,$$

$$\text{or} \quad \tan\frac{\pi}{4n} \tan\frac{5\pi}{4n} \tan\frac{9\pi}{4n} \ldots\ldots \tan\frac{(4n-3)\pi}{4n} = (-1)^m,$$

when n is of the form $2m + 1$.

Hence if n be of the form $4m$, or $4m + 1$, $A = 1$;
if n be of the form $4m + 2$, or $4m + 3$, $A = -1$.

7. $$e^x = 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \ldots\ldots$$

$$\text{and } e^{-x} = 1 - x + \frac{x^2}{1 \cdot 2} - \frac{x^3}{1 \cdot 2 \cdot 3} + \ldots\ldots$$

$$\therefore \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots\ldots$$

Now

$$1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \ldots = \cos x = \left(1 - \frac{2^2x^2}{\pi^2}\right)\left(1 - \frac{2^2x^2}{3^2\pi^2}\right)\left(1 - \frac{2^2x^2}{5^2\pi^2}\right)\ldots.$$

But if the last expression, and

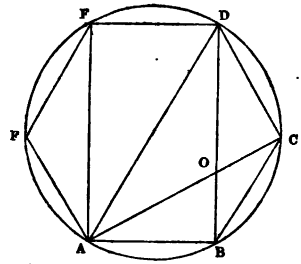
$$\left(1 + \frac{2^2x^2}{\pi^2}\right)\left(1 + \frac{2^2x^2}{3\pi^2}\right)\left(1 + \frac{2^2x^2}{5^2\pi^2}\right)\ldots\ldots$$

be multiplied out, the results will be the same, excepting that in the former the terms will be alternately positive and negative, and in the latter all the terms will be positive,

$$\therefore 2\left\{1 + \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots\ldots\right\}$$

$$= e^x + e^{-x} = 2\left(1 + \frac{2^2x^2}{\pi^2}\right)\left(1 + \frac{2^2x^2}{3^2\pi^2}\right)\ldots\ldots$$

Similarly, $e^x - e^{-x} = 2x\left(1 + \dfrac{x^2}{\pi^2}\right)\left(1 + \dfrac{x^2}{2^2\pi^2}\right)\ldots\ldots$



9. Let $ABCDEF$ be a regular polygon of n sides, r the radius of the circumscribing circle; then the angles subtended at the centre by AB, AC, AD, &c., drawn from A are $\frac{2\pi}{n}$, $\frac{4\pi}{n}$, $\frac{6\pi}{n}$, &c. respectively;

$$\therefore AB = 2r \sin\frac{\pi}{n},$$

$$AC = 2r \sin\frac{2\pi}{n}, \quad AD = 2r\frac{\sin 3\pi}{n}, \text{ \&c.,}$$

and Σ the sum of all these lines

$$= 2r\left\{\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \ldots\ldots + \sin\frac{(n-1)\pi}{n}\right\}.$$

Now let $S = \sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \ldots\ldots + \sin\frac{(n-1)\pi}{n}$;

$$\therefore 2\cos\frac{\pi}{n}\,S = \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \sin\frac{4\pi}{n} + \ldots\ldots + \sin\frac{n\pi}{n}$$

$$+ \sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \ldots\ldots + \sin\frac{(n-2)\pi}{n}$$

$$= S - \sin\frac{\pi}{n} + \sin\frac{n\pi}{n} + S - \sin\frac{(n-1)\pi}{n} = 2S - 2\sin\frac{\pi}{n},$$

hence $S\left(1 - \cos\frac{\pi}{n}\right) = \sin\frac{\pi}{n}$, or $S = \dfrac{\sin\frac{\pi}{n}}{2\sin^2\frac{\pi}{2n}}$;

$$\therefore \Sigma = 2rS = \frac{2r\sin\frac{\pi}{n}}{2\sin^2\frac{\pi}{2n}} = \frac{2a}{2\sin^2\frac{\pi}{2n}} = a\,\mathrm{cosec}^2\frac{\pi}{2n}.$$

Ex. 22.

3. If $x = m\cos\theta$, we have

$$m^3\cos^3\theta - 3m\cos\theta + 1 = 0,$$

$$\text{or } \cos^3\theta - \frac{3}{m^2}\cos\theta + \frac{1}{m^3} = 0 \;\ldots\ldots\ldots\ldots\; (1),$$

$$\text{but } \cos^3\theta - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta = 0 \quad \ldots\ldots\ldots\ldots (2);$$

$\therefore$ comparing equations (1) and (2),

$$m = 2; \quad \text{and } \therefore \cos 3\theta = -\frac{4}{m^3} = -\frac{1}{2},$$

$$\text{or } 3\theta = 120°, \text{ and } 2\pi \pm 120°,$$

hence the values of x are

$$2\cos 40°; \quad 2\cos 80°; \quad -2\cos 20°.$$

5. $x^4 = -1 = \cos(2r+1)\pi \pm \sqrt{-1}\sin(2r+1)\pi;$

$$\therefore x = \cos\frac{(2r+1)\pi}{4} \pm \sqrt{-1}\sin\frac{(2r+1)\pi}{4},$$

and since the index of x is even, we have, giving to r the values 0, 1,

$$x = \cos\frac{\pi}{4} \pm \sqrt{-1}\sin\frac{\pi}{4}, \text{ and } \cos\frac{3\pi}{4} \pm \sqrt{-1}\,\frac{3\pi}{4}$$

$$= \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} \times \sqrt{-1}, \text{ and } -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} \times \sqrt{-1}$$

$$= \frac{1}{2}\{\pm\sqrt{2} \pm \sqrt{-2}\}.$$

Ex. 23.

1. $$x = a^2 + b^2 = a^2\left(1 + \frac{b^2}{a^2}\right),$$

$$\text{put } \frac{b}{a} = \tan\phi, \text{ then}$$

$$x = a^2 \sec^2\phi.$$

5. $$\cos x = \cos A\left\{\cos c \cdot \frac{\sin A}{\cos A}\sin B - \cos B\right\}$$

$$= \cos A\{\cos c \tan A \sin B - \cos B\}.$$

Assume $\quad \cos c \tan A = \cot\phi.$

Then, $\cos x = \cos A \{\cot\phi \sin B - \cos B\}$

$$= \frac{\cos A}{\sin\phi}\{\sin B\cos\phi - \cos B\sin\phi\}$$

$$= \frac{\cos A}{\sin\phi}\sin(B-\phi).$$

Application of Algebra to Geometry.

1. Let x be the greater segment; then

$$12 : x :: x : 12 - x,$$

$$\text{or } x^2 = 144 - 12x;$$

$$\therefore x^2 + 12x = 144;$$

$$\therefore (x+6)^2 = 180;$$

$$\therefore x + 6 = 13{\cdot}4164;$$

$$\therefore x = 7{\cdot}4164 \text{ in.},$$

$$\text{and } 12 - x = 4{\cdot}5836 \text{ in.}$$

2. Let $x =$ the hypothenuse.

$$\text{Then } x^2 = (x-8)^2 + 20^2$$

$$= x^2 - 16x + 464;$$

$$\therefore 16x = 464, \quad x = 29;$$

$$\therefore x - 8 = 21;$$

$\therefore$ the sides are 20, 21, 29.

7. Let $CB = x$,

$CA = z$;

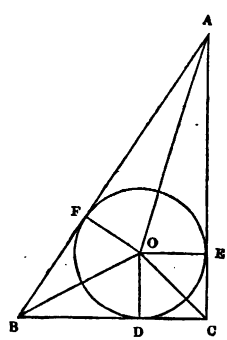
then $AB = (x^2 + z^2)^{\frac{1}{2}}$,

$$x + z + (x^2+z^2)^{\frac{1}{2}} = 20 \ldots\ldots\ldots\ldots (1),$$

$$CB \times CA = 2 \text{ area of } ABC$$

$$= OD \times BC + OE \times CA + OF \times AB;$$

$$\therefore xz = 20 \times \frac{5}{4} = 25 \ldots\ldots\ldots\ldots (2).$$



From (1),

$$x^2 + z^2 = 20^2 - 40(x+z) + (x^2 + z^2 + 2xz);$$

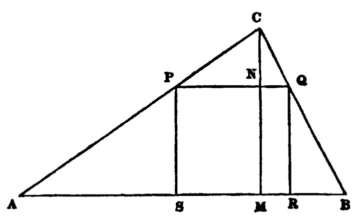
$$\therefore\ x+z=\frac{400+50}{40}=\frac{45}{4}.$$

Square and subtract 4 times (2); then

$$x-z=\sqrt{\left(\frac{45}{4}\right)^2-4\times 25}=\sqrt{\frac{2025-1600}{16}}=\frac{5}{4}\sqrt{17};$$

$$\left.\begin{aligned}\therefore\ x&=\frac{5}{8}(9+\sqrt{17})=8{\cdot}20194,\\ z&=\frac{5}{8}(9-\sqrt{17})=3{\cdot}048,\end{aligned}\right\};\quad \therefore\ AB=20-\frac{45}{4}=8{\cdot}75.$$

9. Let $AB=c$,
$PQ=s$,
$CA=x$,
$CB=z$.



Then $x^2+z^2=c^2$.........(1).

By similar triangles

$$CM : CN = AB : PQ;$$

$$\therefore\ CM : CM-s=c : s.$$

Dividendo,

$$CM : s=c : c-s;\ \ \therefore\ CM=\frac{cs}{c-s}.$$

Now $CA\times CB=2$ area of $ABC=CM\times AB$;

$$\therefore\ xz=\frac{c^2s}{c-s}\text{..............................(2)}.$$

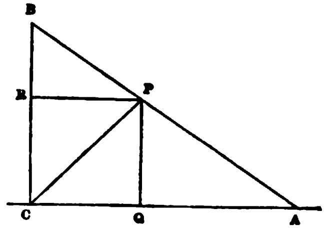
Multiply this equation by (2), and add the result to (1);

$$\text{hence}\quad x+z=c\left(1+\frac{2s}{c-s}\right)^{\frac{1}{2}}=c\left(\frac{c+s}{c-s}\right)^{\frac{1}{2}}.$$

$$\text{Similarly,}\quad x-z=c\left(1-\frac{2s}{c-s}\right)^{\frac{1}{2}}=c\left(\frac{c-3s}{c-s}\right)^{\frac{1}{2}};$$

$$\therefore\ x=\frac{c}{2}\left\{\frac{(c+s)^{\frac{1}{2}}+(c-3s)^{\frac{1}{2}}}{(c-s)^{\frac{1}{2}}}\right\},\qquad z=\frac{c}{2}\left\{\frac{(c+s)^{\frac{1}{2}}-(c-3s)^{\frac{1}{2}}}{(c-s)^{\frac{1}{2}}}\right\}.$$

(2) Using the same notation, we have



$$x^2+z^2=c^2 \quad\ldots\ldots\ldots\ldots\ldots(1),$$

$$xz = 2 \text{ area of } ABC$$

$$= s(x+z)\ldots\ldots\ldots\ldots\ldots(2);$$

$$\therefore (x+z)^2 - 2s(x+z) + s^2 = c^2+s^2,$$

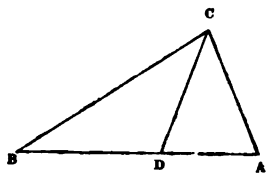
$$x+z = s + (c^2+s^2)^{\frac{1}{2}},$$

$$x-z = \{c^2 - 2s(x+z)\}^{\frac{1}{2}} = \{c^2 - 2s^2 - 2s(c^2+s^2)^{\frac{1}{2}}\}^{\frac{1}{2}}.$$

Hence the sides

$$=\frac{1}{2}\left[s + (c^2+s^2)^{\frac{1}{2}} \pm \{c^2-2s^2-2s(c^2+s^2)^{\frac{1}{2}}\}^{\frac{1}{2}}\right].$$

11. Let S, S_1, S_2 be the areas of the triangles ABC, CBD, ACD respectively; p, p_1, p_2 be the semi-perimeters of the triangles ABC, CBD, ACD respectively.



Let $AB=c$, $AC=b$, $CB=a$.

$$\text{Then } r=\frac{S}{p}, \quad r_1=\frac{S_1}{p_1}, \quad r_2=\frac{S_2}{p_2};$$

$$\therefore \left(\frac{r_1}{r}\right)^2+\left(\frac{r_2}{r}\right)^2=\left(\frac{S_1}{S}\right)^2\left(\frac{p}{p_1}\right)^2+\left(\frac{S_2}{S}\right)^2\left(\frac{p}{p_2}\right)^2$$

$$=\left(\frac{a^2}{c^2}\right)^2\left(\frac{c}{a}\right)^2+\left(\frac{b^2}{c^2}\right)^2\left(\frac{c}{b}\right)^2$$

$$=\frac{a^2+b^2}{c^2}=1;$$

$$\therefore r^2=r_1^2+r_2^2.$$

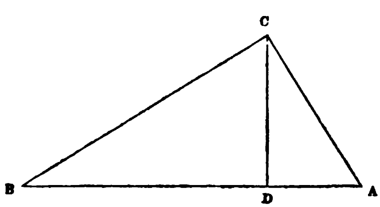
13. Let $CA=5$,

$CB=6{\cdot}4$,

$CD=4$;

$BD=x$,

$DA=y$.



Then $BD : DA :: BC : CA$, by Eucl. Bk. VI. Prop. III.;

$$\therefore x : y = 6{\cdot}4 : 5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

And $BD \times DA + DC^2 = CB \times CA$ by Eucl. Bk. VI. Prop. B.;

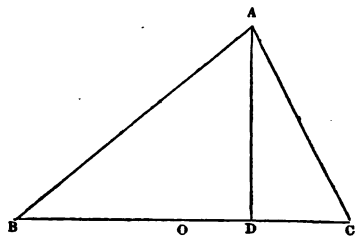
$$\therefore xy = 5 \times 6{\cdot}4 - 4^2 = 16.$$

$$\text{From (1), } \frac{x}{y} = \frac{6{\cdot}4}{5} = \frac{64}{50};$$

$$\therefore x^2 = \frac{16 \times 64}{50}, \quad \therefore x = \frac{32}{10}\sqrt{2} = 3{\cdot}2\sqrt{2},$$

$$y^2 = \frac{16 \times 50}{64}, \quad \therefore y = \frac{20}{8}\sqrt{2} = 2{\cdot}5\sqrt{2};$$

$$\therefore AB = x + y = 5{\cdot}7\sqrt{2} = 8{\cdot}06094.$$



14. Bisect $BC = 14$ in O;

let $OD = x$,

$BA - AC = 3\frac{1}{2}$,

$DA = 8$;

Then

$BA^2 = BD^2 + DA^2 = (7+x)^2 + 8^2$,

$AC^2 = CD^2 + DA^2 = (7-x)^2 + 8^2$;

$$\therefore BA^2 - AC^2 = 28x.$$

$$\text{But } BA - AC = 3{\cdot}5;$$

$$\therefore BA + AC = \frac{28x}{3{\cdot}5} = 8x;$$

$$\therefore BA = 4x + \frac{7}{4}, \text{ and } AC = 4x - \frac{7}{4},$$

$$\left(4x + \frac{7}{4}\right)^2 = (7 + x)^2 + 8^2,$$

$$16x^2 - x^2 = 49 + 64 - \frac{49}{16} = \frac{1759}{16};$$

$$\therefore x = \frac{1}{4}\left(\frac{1759}{15}\right)^{\frac{1}{2}} = \frac{1}{4} \times 10{\cdot}829;$$

$$\therefore AB = 10{\cdot}829 + 1{\cdot}75 = 12{\cdot}579; \quad \therefore AC = 9{\cdot}079.$$

17. In the fig. to (14), let $AB + AC = 2d$, $AD = p$,

$$BD - DC = 2n; \quad BO = OC = x;$$

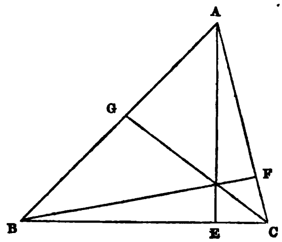
$$\therefore BD = x + n, \quad CD = x - n.$$

$$\text{Then } AB^2 = (x+n)^2 + p^2,$$
$$AC^2 = (x-n)^2 + p^2.$$

Hence as in (14), $AB = d + \frac{nx}{d}$, $AC = d - \frac{nx}{d}$,

and $x = d\left(1 + \frac{p^2}{n^2 - d^2}\right)^{\frac{1}{2}}$, $\therefore$ the sides $= d \pm n\left(1 + \frac{p^2}{n^2 - d^2}\right)^{\frac{1}{2}}$.

20. Let $BC = x$,
$CA = y$,
$AB = z$;
$AE = h$,
$BF = k$,
$CG = l$.
$2s = x + y + z$.



Then, 2 area of $ABC = hx = ky = lz$,

$$s = \frac{1}{2}(x+y+z) = \frac{x}{2}\left(1 + \frac{h}{k} + \frac{h}{l}\right),$$

$$s - x = \frac{1}{2}(-x+y+z) = \frac{x}{2}\left(-1 + \frac{h}{k} + \frac{h}{l}\right),$$

$$s - y = \frac{1}{2}(x-y+z) = \frac{x}{2}\left(1 - \frac{h}{k} + \frac{h}{l}\right),$$

$$s - z = \frac{1}{2}(x+y-z) = \frac{x}{2}\left(1 + \frac{h}{k} - \frac{h}{l}\right).$$

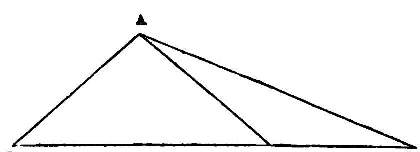
Now area of $ABC = \sqrt{s(s-x)(s-y)(s-z)}$;

$$\therefore \frac{x}{2}h = \frac{x^2}{4(kl)^2}\{(kl+hl+hk)(hk-kl+hl)(kl-hl+hk)(kl+hl-hk)\}^{\frac{1}{2}}.$$

If the root be put $= D$, we obtain

$$x = \frac{2(hkl)^2}{hD}; \quad y = \frac{2(hkl)^2}{kD}; \quad z = \frac{2(hkl)^2}{lD}.$$

25. Let $BC = a$,
$CA = b$,
$AB = c$;
$BP = x$;
$\therefore CP = a - x$.



$$\text{Now } AP^2 = BP \times CP = x(a-x).$$

$$\text{Also } c^2 + x^2 - 2cx \cos B = AP^2 = x(a-x);$$

$$\therefore\ 2x^2 - ax - 2c\left(\frac{a^2+c^2-b^2}{2ac}\right)x + c^2 = 0;$$

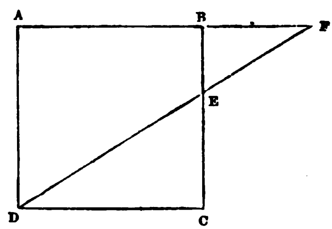
$$\therefore\ x^2 - \frac{x}{2a}(2a^2+c^2-b^2) + \left(\frac{2a^2+c^2-b^2}{4a}\right)^2 = \left(\frac{2a^2+c^2-b^2}{4a}\right)^2 - \frac{c^2}{2};$$

$$\therefore\ x = \frac{1}{4a}\{2a^2+c^2-b^2 \pm (4a^4+b^4+c^4-4a^2b^2-4a^2c^2-2b^2c^2)^{\frac{1}{2}}\}.$$

26. Let $AB = AD = 30$;

$FE = 16$;

$BF = x$, $BE = z$.



Then

$x^2 + z^2 = 16^2 = 256$ .... (1).

By similar triangles,

$$BF : BE :: AF : AD;$$

$$\therefore\ 30x = (30+x)z;$$

$$\therefore\ xz = 30(x-z) \text{ .... } (2).$$

Subtract twice equation (2) from (1),

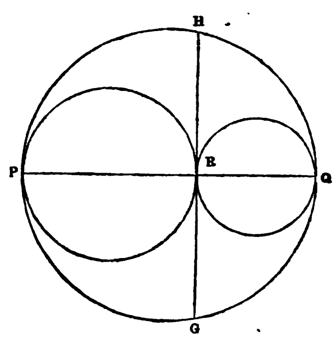
$$\text{hence } (x-z)^2 + 60(x-z) + 30^2 = 900 + 256 = 1156;$$

$$\therefore\ x - z = 34 - 30 = 4;$$

$$\therefore\ x + z = \sqrt{256 + 60 \times 4} = 22{\cdot}2710;$$

$$\therefore\ x = 13{\cdot}1355, \text{ and } z = 9{\cdot}1355.$$

31. Let P, Q, and R be the points of contact of the three circles, and HG the chord of the exterior which touches the two interior circles at their point of contact R; let A, B, C, and E be the areas of the circles described on PQ, PR, RQ, and HG respectively, as diameters; and let



$PR = d$,

$RQ = d$.

Then we have

$$A - B - C = \frac{\pi}{4}(PQ^2 - PR^2 - RQ^2)$$

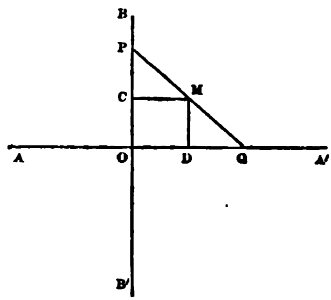
$$= \frac{\pi}{4}\{(d + d')^2 - d^2 - d'^2\}$$

$$= \frac{\pi}{2} dd'.$$

But $E = \pi . HR^2 = \pi . PR . RQ = \pi dd'$.

Hence $A - B - C = \frac{1}{2} E.$

32. Let AA', BB' be the given lines, M the given point, PQ the required line. From M draw MC, MD perpendicular to BB', AA' respectively. Let



$$MD = MC = d,$$

$$DQ = x,$$

$$CP = y,$$

then $PM^2 + MQ^2 = b^2$,

or $d^2 + y^2 + d^2 + x^2 = b^2$,

and from the similar triangles PCM, MDQ,

$$\frac{x}{d} = \frac{d}{y}, \text{ or } xy = d^2;$$

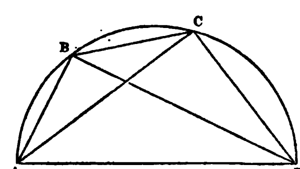
$$\therefore\ x^2 + y^2 + 2xy = b^2; \quad \therefore\ x + y = b,$$

or $x^2 + y^2 - 2xy = b^2 - 4d^2$; $\therefore\ x - y = (b^2 - 4d^2)^{\frac{1}{2}}$;

whence $x = \frac{1}{2}\{b \pm (b^2 - 4d^2)^{\frac{1}{2}}\}$;

$$\therefore\ OQ = \frac{1}{2}\{b + 2d \pm (b^2 - 4d^2)^{\frac{1}{2}}\}.$$

36. Let AB, BC, CD be the chords in the semicircle $ABCD$. Join AC, BD. Let



$$AB = a, \quad BC = b,$$

$$CD = c, \quad AD = x,$$

Then (Eucl. VI. D),

$$AC . BD = AD . BC + AB . CD = bx + ac.$$

But, since ABD, ACD are right angles, we have

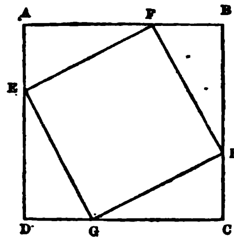
$$AC=(x^2-c^2)^{\frac{1}{2}},\quad BD=(x^2-a^2)^{\frac{1}{2}};$$

$$\therefore\ (x^2-a^2)^{\frac{1}{2}}(x^2-c^2)^{\frac{1}{2}}=bx+ac.$$

Or, squaring both sides, and reducing the result

$$x^3-(a^2+b^2+c^2)\,x-2abc=0.$$

41. Let $ABCD$ be the given square, and $EFHG$ the inscribed square, having its side equal to a given line.



Let $AB=a$, $EF=b$, $AF=x$, and

$$\therefore\ AE=FB=a-x.$$

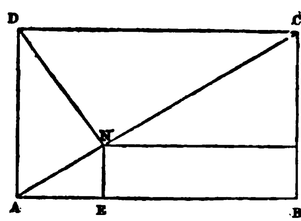
Since $AE^2+AF^2=EF^2$;

$$\therefore\ x^2+(a-x)^2=b^2:$$

whence $x=\frac{1}{2}\{a\pm(2b^2-a^2)^{\frac{1}{2}}\}$;

therefore the limits of b are a and $\dfrac{a}{\sqrt{2}}$.

42. Let $AB=a$, $BC=b$, $AC=d$. Draw DN perpendicular to AC, and let



$$NE=p,\quad NF=q.$$

By similar triangles

$$NE : BC :: AN : AC,$$

also, $AN : AD :: AD : AC$;

$$\therefore\ p=\frac{b}{d}AN=\frac{b}{d}\cdot\frac{b^2}{d}=\frac{b^3}{d^2}.$$

Similarly $q=\dfrac{a^3}{d^2}$;

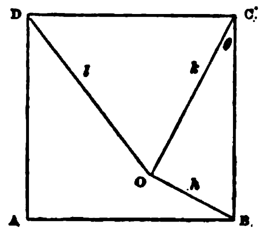
$$\therefore\ p^{\frac{2}{3}}+q^{\frac{2}{3}}=\frac{b^2+a^2}{d^{\frac{4}{3}}}=\frac{d^2}{d^{\frac{4}{3}}}=d^{\frac{2}{3}}.$$

43. Let the angle $OCB = \theta$,

and $\therefore OCD = 90° - \theta$,

then $h^2 = x^2 + k^2 - 2kx \cos\theta$,

$l^2 = x^2 + k^2 - 2kx \sin\theta$;



$\therefore x^2 + k^2 - h^2 = 2kx \cos\theta$ ....(1),

and $x^2 + k^2 - l^2 = 2kx \sin\theta$............(2),

and adding the squares of (1) and (2) we get

$$x^4 + 2(k^2 - h^2)x^2 + (k^2 - h^2)^2 + x^4 + 2(k^2 - l^2)x^2 + (k^2 - l^2)^2 = 4k^2x;$$

$$\therefore 2x^4 - 2(h^2 + l^2)x^2 + (h^2 + l^2)^2 = 2(h^2k^2 + h^2l^2 + k^2l^2 - k^4),$$

$$x = \frac{1}{\sqrt{2}}\{h^2 + l^2 \pm (4h^2k^2 + 4k^2l^2 + 2h^2l^2 - h^4 - l^4 - 4k^4)^{\frac{1}{2}}\}^{\frac{1}{2}}.$$

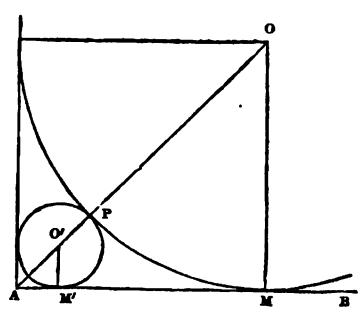
49. It may easily be shewn that

$$r = 2r_1, \quad r_1 = 2r_2, \quad r_2 = 2r_3, \quad \text{\&c.} = \text{\&c.};$$

$$\therefore r + r_1 + r_2 + r_3 + \ldots\ldots = 2(r_1 + r_2 + r_3 + \ldots\ldots);$$

$$\therefore r = r_1 + r_2 + r_3 + \ldots\ldots$$

51. Let r be the radius, and O the centre of any circle touching the two sides of the square terminated at the point A; r_1 the radius, and O' the centre of the circle which touches the last circle, and the same two sides of the square; draw OM, $O'M'$ perpendicular to one of the sides of the square; then $OM = AM$, $O'M' = AM'$; therefore $AO'O$ is a right line passing through P, the point of contact of the two circles,



$$\text{and } AO = r\sqrt{2} = OP + PO' + O'A = r + r_1 + r_1\sqrt{2},$$

$$\text{or } r_1 = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \cdot r = (3 - 2\sqrt{2})\,r.$$

Similarly if r_2, r_3, &c. be the radii of the consecutive circles, we shall have

$$r_2 = (3 - 2\sqrt{2})\,r_1; \quad r_3 = (3 - 2\sqrt{2})\,r_2, \quad \text{\&c.} = \text{\&c.},$$

and the sum of the areas of all the circles

$$= \pi r^2 + \pi r_1^2 + \pi r_2^2 + \pi r_3^2 + \ldots\ldots$$

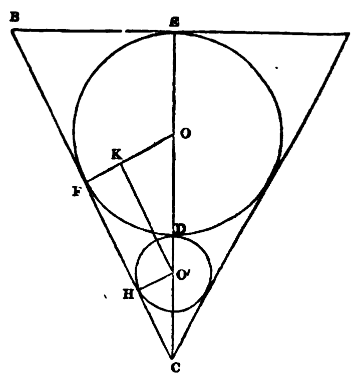
$$= \pi r^2 \left\{1 + \left(\frac{r_1}{r}\right)^2 + \left(\frac{r_2}{r}\right)^2 + \left(\frac{r_3}{r}\right)^2 + \ldots\ldots\right\}$$

$$= \frac{\pi r^2}{1 - \left(\frac{r_1}{r}\right)^2} = \frac{\pi r^2}{1 - (17 - 12\sqrt{2})} = \pi r^2 \frac{1}{12\sqrt{2} - 16}$$

$$= \frac{\pi a^2}{4}\left(\frac{12\sqrt{2} + 16}{288 - 256}\right) = \frac{\pi a^2}{4}\left(\frac{3\sqrt{2} + 4}{8}\right),$$

if a be a side of the square.

53. Let O, O' be the centres of the spheres, touching the slant sides of the cone in the points F, H.



Let $OE = OF = R$;

$O'D = O'H = r$.

From similar triangles, we have

$$R : r :: OC : O'C,$$

or $R = r : r :: R + r : O'C$;

$$\therefore\ O'C = \frac{r(R+r)}{R-r}.$$

Similarly, $OC = \dfrac{R(R+r)}{R-r}$;

$$\therefore\ EC = \frac{Rr + r^2}{R-r} + 2R + r = \frac{2R^2}{R-r}.$$

Again $BE : BC :: OK : OO' :: R - r : R + r$;

$$\therefore\ BE^2 : EC^2 = (R-r)^2 : 4Rr;$$

$$\therefore\ BE^2 = \frac{4R^4}{(R-r)^2} \times \frac{(R-r)^2}{4Rr} = \frac{R^3}{r}.$$

Volume of cone $= \dfrac{1}{3}\pi BE^2 \times EC = \dfrac{\pi}{3}.\dfrac{R^3}{r}.\dfrac{2R^2}{R-r} = \dfrac{2\pi}{3}\left(\dfrac{R^5}{Rr - r^2}\right).$

54. Referring to the figure of the last example, let O, O_1, O_2, &c. be the centres of the spheres; then r, r_1, r_2, &c. being the radii of the spheres, we have

$$r = OC \sin\alpha = (r + r_1 + O_1C)\sin\alpha = (r + r_1)\sin\alpha + r_1;$$

$$\therefore\ r_1 = r\left(\frac{1 - \sin\alpha}{1 + \sin\alpha}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

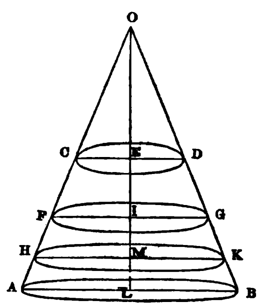
Again, $r_1 = O_1C\sin\alpha = (r_1 + r_2 + O_2C)\sin\alpha = (r_1 + r_2)\sin\alpha + r_2$;

$$\therefore\ r_2 = r_1\left(\frac{1 - \sin\alpha}{1 + \sin\alpha}\right) = r\left(\frac{1 - \sin\alpha}{1 + \sin\alpha}\right)^2 \ldots\ldots\ldots (2).$$

Similarly, $r_3 = r\left(\frac{1 - \sin\alpha}{1 + \sin\alpha}\right)^3$, &c. = &c.,

or the radii are in geometrical progression.

55. Let $ACDB$ be the given frustum of a cone. Complete the cone. Let FG, HK be the diameters of the sections required. Draw the altitude OL of the cone, meeting the sections in E, I, M and L. Let



$$AL = a,$$
$$CE = b,$$
$$EL = h,$$
and $FI = x$.

Then we have

$$a : b :: OE + h : OE;$$

$$\therefore\ a - b : b :: h : OE; \quad \therefore\ OE = \frac{bh}{a - b}.$$

Again, $x : b :: EI + \frac{bh}{a-b} : \frac{bh}{a-b}$;

$$\therefore\ x - b : b :: EI : \frac{bh}{a - b};$$

$$\therefore\ EI = \frac{h(x - b)}{a - b};$$

therefore, by the question, we have, since the volume of the given frustum $= \frac{\pi h}{3}(a^2 + ab + b^2)$,

$$\frac{\pi h(x-b)}{3(a-b)}(x^2+bx+b^2)=\frac{\pi h}{9}(a^2+ab+b^2);$$

$$\therefore\ 3(x^3-b^3)=a^3-b^3,$$

whence $x=\left(\dfrac{a^3+2b^3}{3}\right)^{\frac{1}{3}}$ = the radius of one section.

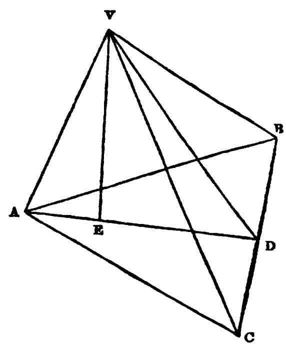
Similarly, the radius of the other section

$$=\left(\frac{2a^3+b^3}{3}\right)^{\frac{1}{3}}.$$

56. Let $AB=AC=12,$

$$=VB=VC,$$

$$BC=10=VA.$$



Then

$AD^2=AC^2-CD^2=12^2-5^2=119;$

$\therefore$ area of base $ABC=\frac{1}{2}10\sqrt{119}.$

In the triangle VAD,

$VD^2=VA^2+AD^2-2AD\times AE;$

$$\therefore\ AE=\frac{VA^2}{2AD}=\frac{100}{2\sqrt{119}},\ \text{since}\ VD=AD.$$

$$VE^2=VA^2-AE^2=100-\frac{100^2}{4\times 119}=100\left(1-\frac{25}{119}\right).$$

Content of pyramid $=\frac{1}{3}VE\times ABC=\frac{1}{3}\left(10\sqrt{\frac{94}{119}}\right)(5\sqrt{119})$

$$=\frac{50}{3}\sqrt{94}=161{\cdot}589.$$

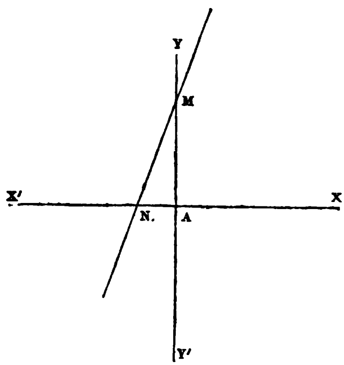
ANALYTICAL GEOMETRY

AND

CONIC SECTIONS.

I. Straight Line.

Ex. 1.



1. If $y=0$,

$$x=-\frac{1}{3},$$

and if $x=0$,

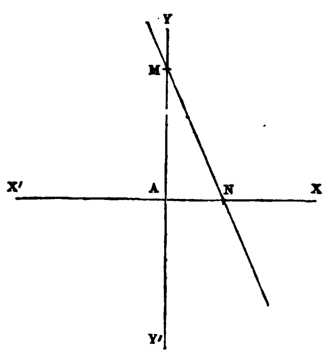
$$y=1.$$

Therefore, $X'AX$ and YAY' being assumed as co-ordinate axes. Take

$$AN=\frac{1}{3},$$

and $AM=1$,

and join MN, which will be the line required.



5. Dividing the given equation by 2, we have

$$\frac{x}{6}+\frac{y}{14}=1.$$

On the rectangular axes $X'AX$, YAY', take

$$AN=6,$$

and $AM=14$,

and join NM,

MN will be the line required.

9. At the point of intersection of the lines, the values of x and y will be the same in both equations,

$$6y - 2x = 0,$$

$$y + 2x = 1;$$

$$\therefore\ 7y = 1, \text{ and } y = \frac{1}{7},$$

$$x = 3y = \frac{3}{7}.$$

13. Let AD be the distance from the origin of co-ordinates, of the given line, and let $\angle DAX = \alpha$, then

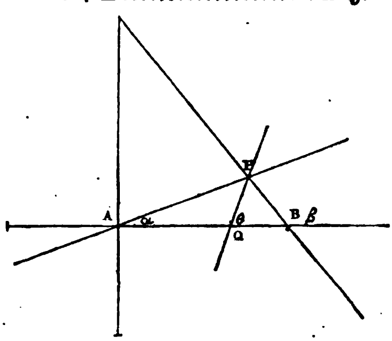
$$\cot\alpha = \frac{2}{5}, \text{ and therefore } \cos\alpha = \frac{2}{\sqrt{29}}.$$

$$\text{Now } AD = x\cos\alpha = 5 \times \frac{2}{\sqrt{29}} = \frac{10}{\sqrt{29}}.$$

18. Let $y = \frac{2}{5}x$ be the equation to AP,

$$y = -\frac{4}{3}x + 4 \ldots\ldots\ldots\ldots\ldots\ldots BP,$$

$$y = Ax + B \ldots\ldots\ldots\ldots\ldots\ldots\ldots PQ.$$



The co-ordinates of the point of intersection P are

$$x' = \frac{30}{13}, \quad y' = \frac{12}{13}.$$

Hence, $y - y' = A(x - x')$.

Since PQ bisects the angle APB,

$$\theta = \alpha + \frac{\beta - \alpha}{2} = \frac{\beta + \alpha}{2};$$

$$\therefore \tan 2\theta = \frac{\tan\beta + \tan\alpha}{1 - \tan\beta\tan\alpha} = \frac{-\frac{4}{3}+\frac{2}{5}}{1-\left(-\frac{4}{3}\right)\frac{2}{5}} = -\frac{14}{23}$$

$$= \tan(180° - 31° \; 19' \; 43'');$$

$$\therefore A \text{ or } \tan\theta = \tan 74° \; 20' \; 8'' = 3{\cdot}566;$$

$$\therefore y - \frac{12}{13} = 3{\cdot}566\left(x - \frac{30}{13}\right).$$

N.B. If $\tan\alpha = a$, $\tan\beta = m$,

$$\tan 2\theta = \frac{m+a}{1-ma} = \frac{1}{c} \text{ suppose};$$

$$\therefore 2c\tan\theta = 1 - \tan^2\theta,$$

$$\tan^2\theta + 2c\tan\theta + c^2 = 1 + c^2;$$

$$\therefore \tan\theta = -c \pm \sqrt{1+c^2} = \frac{-(1-ma) \pm \sqrt{1+m^2+a^2+m^2a^2}}{m+a};$$

$$\therefore y - y' = \frac{ma - 1 \pm \sqrt{(1+m^2)(1+a^2)}}{m+a}(x - x').$$

22. $y^2 - 2xy\sec\theta + x^2\sec^2\theta = x^2\sec^2\theta - x^2 = x^2\tan^2\theta;$

$$\therefore y = x(\sec\theta \pm \tan\theta),$$

the equations to two straight lines passing through the origin. If ϕ be the $\angle$ between them,

$$\tan\phi = \frac{(\sec\theta + \tan\theta) - (\sec\theta - \tan\theta)}{1 + (\sec^2\theta - \tan^2\theta)} = \frac{2\tan\theta}{2};$$

$$\therefore \phi = \theta.$$

25_1. The equation may be written thus,

$$4x(3y+2) - 9(3y+2) = 0,$$

$$(4x-9)(3y+2) = 0;$$

$\therefore 4x - 9 = 0$, $x = \frac{9}{4}$ the equation to a line parallel to the axis of y,

$3y + 2 = 0$, $y = -\frac{2}{3}$, ... x.

25. Solving the equation with regard to y,

$$y^2 + 2y(2x-1) + (2x-1)^2 = (4x^2 - 4x + 1) - x^2 - 2x + 2 = 3(x^2 - 2x + 1);$$

$$\therefore\ y = -(2x-1) \pm (x-1)\sqrt{3} = (-2 \pm \sqrt{3})x + (1 \mp \sqrt{3}),$$

which represents two straight lines.

27. The equations of two straight lines referred to oblique axes inclined at an angle ω, being $y - mx = 0$, $my + x = 0$; find the angle between them.

Let θ be the angle between the lines, and α, β, their respective inclinations to the axis of x; then

$$\frac{\sin\alpha}{\sin(\omega-\alpha)} = m \quad\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

$$\text{and}\quad \frac{\sin\beta}{\sin(\omega-\beta)} = -\frac{1}{m} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\text{whence}\quad \tan\alpha = \frac{m\sin\omega}{1+m\cos\omega} \quad\ldots\ldots\ldots\ldots (3),$$

$$\tan\beta = -\frac{\sin\omega}{m-\cos\omega} \quad\ldots\ldots\ldots\ldots (4).$$

Now $\theta = \alpha - \beta$, and

$$\tan\theta = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta}$$

$$= \frac{\dfrac{m\sin\omega}{1+m\cos\omega} + \dfrac{\sin\omega}{m-\cos\omega}}{1 - \dfrac{m\sin\omega}{1+m\cos\omega} \cdot \dfrac{\sin\omega}{m-\cos\omega}} = \frac{(m^2+1)\sin\omega}{(m^2-1)\cos\omega}$$

$$= \frac{m^2+1}{m^2-1}\tan\omega.$$

30. Let $r\cos(\theta-\alpha) = p$ be the equation to the line. Then, since the line passes through the two points $(r_1\alpha_1)$, $(r_2\alpha_2)$ we have

$$r_1\cos(\alpha_1 - \alpha) = p \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\text{and}\quad r_2\cos(\alpha_2 - \alpha) = p \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (1) and (2) we get

$$(r\cos\theta - r_1\cos\alpha_1)\cos\alpha + (r\sin\theta - r_1\sin\alpha_1)\sin\alpha = 0,$$

and from (2) and (3)

$$(r_1 \cos \alpha_1 - r_2 \cos \alpha_2) \cos \alpha + (r_1 \sin \alpha_1 - r_2 \sin \alpha_2) \sin \alpha = 0;$$

$$\therefore \frac{r \cos \theta - r_1 \cos \alpha_1}{r_1 \cos \alpha_1 - r_2 \cos \alpha_2} = \frac{r \sin \theta - r_1 \sin \alpha_1}{r_1 \sin \alpha_1 - r_2 \sin \alpha_2}.$$

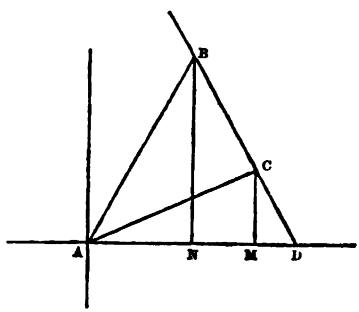
Whence, by reduction, we obtain

$$\frac{\sin (\alpha_2 - \alpha_1)}{r} + \frac{\sin (\theta - \alpha_2)}{r_1} + \frac{\sin (\alpha_1 - \theta)}{r_2} = 0.$$

36. Let $y = x \tan \alpha$, be the equation to AB ....(1),

$y = x \tan \alpha_1$ .... AC ....(2),

$y = x \tan \alpha_2 + a$..................... BCD ......(3).



Now $\Delta ABC = \Delta ABD - \Delta ACD = \frac{1}{2} AD (BN - CM)$.

In (3) put $y = 0$; then x or $AD = -\dfrac{a}{\tan \alpha_2}$.

To find BN, eliminate x between (1) and (3); then

$$y (\tan \alpha - \tan \alpha_2) = a \tan \alpha;$$

$$\therefore y \text{ or } BN = \frac{a \tan \alpha}{\tan \alpha - \tan \alpha_2} = \frac{a \sin \alpha \cos \alpha_2}{\sin (\alpha - \alpha_2)}.$$

$$\text{Similarly, } CM = \frac{a \sin \alpha_1 \cos \alpha_2}{\sin (\alpha_1 - \alpha_2)};$$

$$\therefore \Delta ABC = -\frac{a^2 \cos^2 \alpha_2}{2 \sin \alpha_2} \left\{ \frac{\sin \alpha}{\sin (\alpha - \alpha_2)} - \frac{\sin \alpha_1}{\sin (\alpha_1 - \alpha_2)} \right\},$$

which by reduction,

$$= \frac{a^2}{2} \left\{ \frac{\sin (\alpha - \alpha_1) \cos^2 \alpha_2}{\sin (\alpha - \alpha_2) \sin (\alpha_1 - \alpha_2)} \right\}$$

39. Draw the ordinates AL, BM, CN from the angular points A, B, C of the triangle, whose abscissæ, a, a', a'' are in order of magnitude; then the area of ABC = trapezoid $ABML$ + trapezoid $BCNM$ − trapezoid $ACNL$

$$=\frac{1}{2}\{(a'-a)(b+b')+(a''-a')(b'+b'')-(a''-a)(b+b'')\}.$$

II. Circle.

Ex. 2.

1. The equation may be put under the form

$$x^2+4x+4+y^2-6y+9=4+9+3=16,$$
$$\therefore (x+2)^2+(y-3)^2=4^2,$$

hence, $h=-2$, $k=3$; and $c=4$.

6. Comparing

$$r^2-2(\cos\theta+\sqrt{3}\sin\theta)\,r-5=0,$$

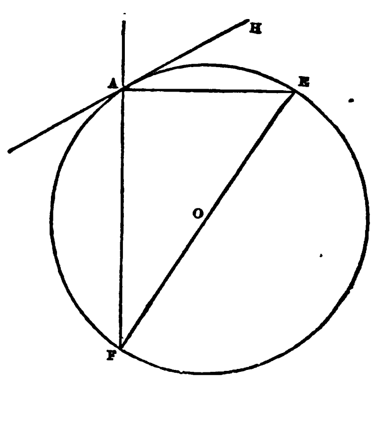
with $r^2-2r(h\cos\theta+k\sin\theta)+h^2+k^2-c^2=0$,

the general polar equation to the circle, we have

$$h\cos\theta=\cos\theta;\quad \therefore h=1,$$
$$k\sin\theta=\sqrt{3}\sin\theta;\quad \therefore k=\sqrt{3},$$
$$\text{and } k^2+h^2-c^2=-5;\quad \therefore c=3.$$

For the centre,

$$\tan\theta=\frac{k}{h}=\sqrt{3};\quad \therefore \theta=\frac{1}{3}\pi,$$
$$\text{and } r^2=h^2+k^2=4;\quad \therefore r=2.$$



11. Let $y=mx$, be the equation to the straight line. The origin of co-ordinates, when the equation to the circle is

$$x^2+y^2-3x+4y=0,$$

is a point in the circumference.

Now when $y=0$,

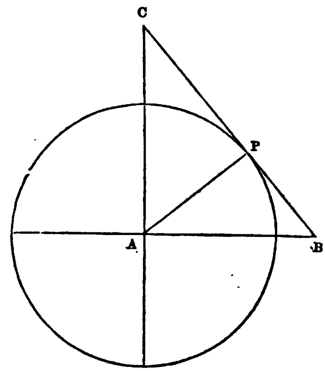
$$x=3=AE,$$
$$x=0,$$
$$y=-4=AF;$$

$$\therefore\ m = \tan EAH = \tan EFA = \frac{3}{4};$$

$\therefore\ 3x = 4y$ is the required equation.

14. Let BPC be the tangent at P whose equation is

$$\frac{x}{a} + \frac{y}{b} = 1;$$



then $AB = a$,

$AC = b$;

radius $= c$.

Now $BP = \dfrac{a^2}{CB}$,

$CP = \dfrac{b^2}{CB}$,

and $BP \times CP = AP^2$;

$$\therefore\ \frac{a^2b^2}{a^2+b^2} = c^2, \text{ or } \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}.$$

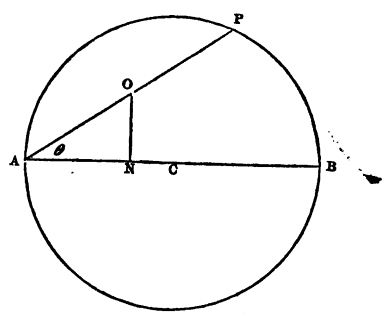
21. If A be the origin,

$$y^2 = 2rx - x^2,$$

is the equation to the circle;

$$y = mx$$

is the equation to the chord AP.



Now $AP = AB\cos\theta = \dfrac{2r}{(1+m^2)^{\frac{1}{2}}}$.

$AN = AO\cos\theta = \dfrac{r}{1+m^2}$,

$$NO = AN\tan\theta = \frac{mr}{1+m^2}.$$

Hence $\left(x - \dfrac{r}{1+m^2}\right)^2 + \left(y - \dfrac{mr}{1+m^2}\right)^2 = \dfrac{r^2}{1+m^2}$;

and $\therefore\ (1+m^2)(x^2+y^2) - 2r(x+my) = 0$,

is the required equation.

23. Take the point in the original circle for pole, and let d be the diameter of this circle; then $r = d\cos\theta$ is its polar equation.

If now the diameter d' of one of the other circles be inclined to the initial axis at an angle α, $d' = d\cos\alpha$, and

$$r = d\cos\alpha\cos(\theta-\alpha)$$

is the equation to this circle.

In like manner, the equations to the other two circles are

$$r = d\cos\beta\cos(\theta-\beta), \text{ and } r = d\cos\gamma\cos(\theta-\gamma).$$

At the point of intersection of the first and second small circles, we have

$$\cos\alpha \, . \cos(\theta-\alpha) = \cos\beta\cos(\theta-\beta);$$

$$\therefore \cos\theta + \cos(\theta - 2\alpha) = \cos\theta + \cos(\theta-2\beta),$$

$$\theta - 2\alpha = \pm(\theta - 2\beta),$$

$$\text{hence } \theta = \alpha+\beta, \text{ and } r = d\cos\alpha\cos\beta.$$

At the point of intersection of the first and third small circles

$$\theta = \alpha+\gamma, \text{ and } r = d\cos\alpha\cos\gamma.$$

In the polar equation to a right line, $r\cos(\lambda-\theta) = p$, substitute these values of θ and r; thence we obtain

$$d\cos\alpha\cos\beta\cos\{\lambda-(\alpha+\beta)\} = p = d\cos\alpha\cos\gamma\cos\{\lambda-(\alpha+\gamma)\};$$

$$\therefore \lambda = \alpha+\beta+\gamma, \text{ and } p = d\cos\alpha\cos\beta\cos\gamma.$$

Hence it appears from the symmetry of these values that the same straight line passes through the intersection of the second and third circles as through the intersections of the first and second, and the first and third.

27. The co-ordinates of the centre of the circle being h, k, the square of the distance of the centre of the circle from the line $x\cos\alpha + y\sin\alpha = a$ will be equal to

$$\frac{(h\cos\alpha + k\sin\alpha - a)^2}{\cos^2\alpha+\sin^2\alpha} = (h\cos\alpha + k\sin\alpha - a)^2.$$

But this distance, the line being a tangent, must be equal to the radius; hence

$$\pm(h\cos\alpha + k\sin\alpha - a) = r \ldots\ldots\ldots\ldots\ldots(1).$$

$$\text{Similarly, } (h\cos\beta + k\sin\beta - b) = r \ldots\ldots\ldots\ldots\ldots(2),$$

$$\text{and } (h\cos\gamma + k\sin\gamma - c) = r \ldots\ldots\ldots\ldots\ldots(3).$$

from equations (1), (2), and (3), we find

$$h=\frac{b(\sin\alpha-\sin\gamma)-a(\sin\beta-\sin\gamma)-c(\sin\alpha-\sin\beta)}{\sin(\alpha-\beta)-\sin(\alpha-\gamma)+\sin(\beta-\gamma)},$$

$$k=\frac{a(\cos\beta-\cos\gamma)-b(\cos\alpha-\cos\gamma)+c(\cos\alpha-\cos\beta)}{\sin(\alpha-\beta)-\sin(\alpha-\gamma)+\sin(\beta-\gamma)}.$$

Substituting these values in (1) we have

$$r=\frac{a\sin(\beta-\gamma)+b\sin(\gamma-\alpha)+c\sin(\alpha-\beta)}{\sin(\alpha-\beta)+\sin(\gamma-\alpha)+\sin(\beta-\gamma)}$$

$$=\frac{a\sin(\beta-\gamma)+b\sin(\gamma-\alpha)+c\sin(\alpha-\beta)}{4\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)\sin\frac{1}{2}(\alpha-\beta)}.$$

29. Compare $x^2-xy+y^2-ax-ay-0$, with the general equation

$$(x-h)^2+(y-k)^2+2(x-h)(y-k)\cos\omega=c^2,$$

where ω is the inclination of the axes, and c the radius.

$$2\cos\omega=-1,\quad\therefore\ \cos\omega=-\frac{1}{2}=\cos\frac{2\pi}{3},$$

$$2h+2k\cos\omega=a,\quad 2k+2h\cos\omega=a,$$

$$\text{and } h^2+k^2+2hk\cos\omega-e^2=0;$$

$$\therefore\ 2h-k=a,\quad 2k-h=a;$$

$$\therefore\ h=a=k,$$

$$c^2=2a^2-a^2=a^2;\quad\therefore\ \text{radius}=a.$$

PARABOLA.

Ex. 3.

1. $$1+2x+3y^2=0;$$

$$\therefore\ y^2=-\frac{2}{3}\left(x+\frac{1}{2}\right),$$

is the equation to a parabola whose latus rectum $=\frac{2}{3}$; co-ordi-

nates of vertex, $h = -\frac{1}{2}$; $k = 0$; the axis of the curve coincides with the axis of x; and the concavity of the parabola is towards the negative direction of x.

4. $$y^2 + 2xy + x^2 + y - 3x + 1 = 0.$$

To take away the term involving xy, let

$$x = t\cos\theta - u\sin\theta, \quad y = t\sin\theta + u\cos\theta.$$

Then

$$\begin{aligned}
y^2 &= t^2\sin^2\theta + 2tu\sin\theta\cos\theta + u^2\cos^2\theta \\
+2xy &= 2t^2\sin\theta\cos\theta + 2tu(\cos^2\theta - \sin^2\theta) - 2u^2\sin\theta\cos\theta \\
+x^2 &= t^2\cos^2\theta - 2tu\sin\theta\cos\theta + u^2\sin^2\theta \\
+y &= \qquad\qquad\qquad\qquad + t\sin\theta + u\cos\theta \\
-3x &= \qquad\qquad\qquad\qquad - 3t\cos\theta + 3u\sin\theta \\
+1 &= \qquad\qquad\qquad\qquad\qquad\qquad + 1;
\end{aligned}$$

$$\therefore\ 0 = t^2(1 + \sin 2\theta) + 2tu\cos 2\theta + u^2(1 - \sin 2\theta)$$
$$+ t(\sin\theta - 3\cos\theta) + u(\cos\theta + 3\sin\theta) + 1.$$

Let the coefficient of $tu = 0$, then $\cos 2\theta = 0 = \cos 90°$;

$$\therefore\ \theta = 45°.$$

Hence $2t^2 + 2\sqrt{2}u - \sqrt{2}t + 1 = 0$,

$$t^2 - \frac{\sqrt{2}}{2}t + \frac{1}{8} = -\sqrt{2}u - \frac{1}{2} + \frac{1}{8} = -\sqrt{2}u - \frac{3}{8};$$

$$\therefore\ \left(t - \frac{1}{2\sqrt{2}}\right)^2 = -\sqrt{2}\left(u + \frac{3\sqrt{2}}{16}\right),$$

the equation to a parabola, of which

$$L = \sqrt{2}, \quad h = \frac{\sqrt{2}}{4}, \quad k = -\frac{3\sqrt{2}}{16};$$

the axis of the curve is parallel to the axis of u, and its concavity is towards the negative direction of the axis of u.

7. $$(y + c)^{\frac{1}{2}} + (x + c)^{\frac{1}{2}} = 2c^{\frac{1}{2}};$$

$$\therefore\ y + x - 2c = -2(y + c)^{\frac{1}{2}}.(x + c)^{\frac{1}{2}};$$

$$\therefore\ y^2 - 2xy + x^2 - 8cy - 8cx = 0.$$

Now proceed as above in (4).

9. The equations to the tangents in terms of their inclinations to the axes, are

$$y = mx + \frac{a}{m} \dots\dots\dots\dots\dots\dots(1),$$

$$\text{and } y = m'x + \frac{a}{m'} \dots\dots\dots\dots\dots\dots(2),$$

and since they are drawn from the same point in the directrix, the values of y in (1) and (2) must be equal, when $x = -a$;

$$\therefore\ -ma + \frac{a}{m} = -m'a + \frac{a}{m'},$$

$$m' - m = -\frac{m' - m}{mm'};$$

$$\therefore\ mm' = -1, \text{ or } m' = -\frac{1}{m};$$

$\therefore$ the tangents are at right angles to each other.

11. Let QRq, PAp be the parabolas; Qq being a tangent to the interior parabola at any point $P(x', y')$.

The equation to the interior parabola being

$$y^2 = 4ax \dots\dots\dots\dots\dots\dots(1),$$

the equation to the exterior one will be of the form

$$y^2 = 4a(x + h) \dots\dots\dots\dots\dots\dots(2).$$

Also the equation to the tangent to (1) is

$$y = \frac{2a}{y'}(x + x') \dots\dots\dots\dots\dots(3).$$

Eliminating x between (3) and (2), we have,

$$y^2 - 2y'y + 4a(x' - h) = 0,$$

the roots of this equation are the ordinates QM, qm, therefore by the Theory of Equations

$$\frac{QM + qm}{2} = y',$$

the ordinate of the point of contact of the tangent Qq, therefore the tangent Qq is bisected at the point of contact.

13. Let $y^2 = lx$ be the equation to the parabola,

and $(x-h)^2 + (y-k)^2 = r^2$ .... circle.

Eliminating x between these equations, we get

$$\left(\frac{y^2}{l} - h\right)^2 + (y-k)^2 = r^2,$$

$$\text{or } y^4 - (2hl - l^2)y^2 - 2l^2ky + l^2(h^2 + k^2 - r^2) = 0.$$

Now the roots of this equation are the ordinates of the points of intersection of the curves; if therefore y_1, y_2, be the ordinates on the upper side of the axis, and $-y_3$, $-y_4$ the ordinates on the lower side, then by the Theory of Equations

$$(y_1 + y_2) - (y_3 + y_4) = 0,$$

$$\text{or } y_1 + y_2 = y_3 + y_4.$$

16. The equation to the tangent to the curve

$$y^2 = 4ax,$$

$$\text{is } y = mx + \frac{a}{m}, \text{ where } m = \tan\theta,$$

and the equation to the tangent to the circle is

$$y = mx + c(1 + m^2)^{\frac{1}{2}},$$

and these two equations represent the same line;

$$\therefore\ c(1 + m^2)^{\frac{1}{2}} = \frac{a}{m},$$

$$\text{or } c.\sec\theta = a\cot\theta,$$

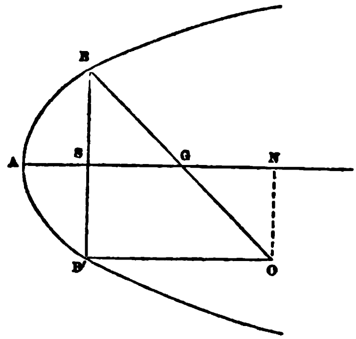
$$\text{hence, } \sin\theta = \frac{-c \pm (c^2 + 4a^2)^{\frac{1}{2}}}{2a}.$$

19. Let $(x-h)^2 + (y-k)^2 = c^2$,

be the equation to the circle of curvature required,

$$\text{radius of curvature} = \frac{BG^3}{SB^2} = \frac{(SB^2 + SG^2)^{\frac{3}{2}}}{SB^2} = 4\sqrt{2a}.$$

Since $SG = 2a = SB$; $\therefore\ B'O = B'B = 4a$;



$\therefore BO = 4\sqrt{2}a$ and O is the centre of the circle of curvature; hence,

$$h = AN = a + 2a + 2a = 5a, \quad k = NO = -2a;$$

$$\therefore (x-5a)^2 + (y+2a)^2 = (4\sqrt{2}a)^2;$$

$\therefore x^2 + y^2 - 10ax + 4ay - 3a^2 = 0$ is the required equation.

22. The equation of either tangent in terms of its inclination to the axis, is of the form

$$y = \beta x + \frac{a}{\beta}, \text{ where } \beta = \tan\theta;$$

$$\therefore \frac{1}{\beta^2} - \frac{y}{a}\cdot\frac{1}{\beta} + \frac{x}{a} = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

but when $\sin\theta \cdot \sin\theta' = m$, $\operatorname{cosec}^2\theta \cdot \operatorname{cosec}^2\theta' = \frac{1}{m^2}$;

$$\therefore \frac{1}{m^2} = \left(\frac{1}{\beta^2} + 1\right)\left(\frac{1}{\beta'^2} + 1\right)$$

$$= \frac{1}{\beta^2} + \frac{1}{\beta'^2} + \frac{1}{\beta^2\beta'^2} + 1$$

$$= \left(\frac{1}{\beta} + \frac{1}{\beta'}\right)^2 + \left(\frac{1}{\beta\beta'} - 1\right)^2;$$

now $\frac{1}{\beta} + \frac{1}{\beta'} = \frac{y}{a}$, and $\frac{1}{\beta\beta'} = \frac{x}{a}$;

$$\therefore \left(\frac{y}{a}\right)^2 + \left(\frac{x-a}{a}\right)^2 = \frac{1}{m^2},$$

$$\text{or}\ \ (x-a)^2+y^2=\frac{a^2}{m^2},$$

which is the equation to a circle whose centre is the focus, and radius $=\dfrac{a}{m}=a\operatorname{cosec}\theta\,.\operatorname{cosec}\theta'$.

(2°). From (1) we have

$$\frac{1}{\beta}=\frac{y}{2a}+\frac{\{y^2-4ax\}^{\frac{1}{2}}}{2a},\ \text{and}\ \frac{1}{\beta'}=\frac{y}{2a}-\frac{(y^2-4ax)^{\frac{1}{2}}}{2a};$$

$$\therefore\ \frac{1}{\beta}-\frac{1}{\beta'}=\cot\theta-\cot\theta'=\frac{(y^2-4ax)^{\frac{1}{2}}}{a}=n;$$

$$\therefore\ y^2-4ax=a^2n^2,$$

$$\text{or}\ \ y^2=4ax+a^2n^2,$$

which is the equation to a parabola,

$$L=4a,\quad h=-\frac{1}{4}an^2,\quad k=0.$$

24. Let $\angle ASP=\phi$, then the polar equation to PT is

$$\frac{2a}{r}=\cos\theta+\cos(\theta-\phi)\ldots\ldots\ldots\ldots\ldots(1).$$

Similarly, if $\angle ASQ=\lambda$,

$$\frac{2a}{r}=\cos\theta+\cos(\theta-\lambda)\ \text{is the equation to}\ QT\ldots\ldots(2).$$

At the point of intersection

$$\cos(\theta-\phi)=\cos(\theta-\lambda);$$

$$\therefore\ \theta-\phi=-(\theta-\lambda);\quad \therefore\ \theta=\frac{1}{2}(\phi+\lambda)\ldots\ldots\ldots\ldots(3);$$

$$\therefore\ \frac{2a}{ST}=\cos\left(\frac{\phi+\lambda}{2}\right)+\cos\left(\frac{\phi-\lambda}{2}\right)=2\cos\frac{\phi}{2}\cos\frac{\lambda}{2},$$

$$\text{or}\ \ ST=\frac{a}{\cos\frac{\phi}{2}\cos\frac{\lambda}{2}},$$

whence, by the polar equation to the parabola

$$ST^2=\frac{a}{\cos^2\frac{\phi}{2}}\cdot\frac{a}{\cos^2\frac{\lambda}{2}}=SP\,.\,SQ.$$

Again, from (3), $\phi + \lambda = 2\theta$,

but $\lambda - \phi = \alpha$ by hypothesis;

$$\therefore\ \theta - \phi = \frac{1}{2}\alpha,$$

which being substituted in (1), we have

$$\frac{2a}{r} = \cos\theta + \cos\frac{1}{2}\alpha;$$

$$\therefore\ r = \frac{2a\sec\frac{1}{2}\alpha}{1 + \sec\frac{1}{2}\alpha\,.\cos\theta},$$

for the required locus, which is an hyperbola, whose eccentricity $= \sec\frac{1}{2}\alpha$.

29. Let β be the constant angle between the tangent and the focal line. The equation to the tangent at any point x', y' is

$$y = \frac{2a}{y'}(x + x') \text{ .... } (1).$$

The equation to the focal line is

$$y = \frac{\frac{2a}{y'} + \tan\beta}{1 - \frac{2a}{y'}\tan\beta}\,.\,(x - a) = \frac{2a + y'\tan\beta}{y' - 2a\tan\beta}(x - a) \text{ .... } (2).$$

Eliminating y', x' by means of (1) and (2) and the relation $y'^2 = 4ax'$, we have

$$y\{y - (x - a)\tan\beta\}\{x - a + y\tan\beta\} - x\{y - (x - a)\tan\beta\}^2$$
$$- a\{x - a + y\tan\beta\}^2 = 0 \text{......} (3),$$

whence, reducing,

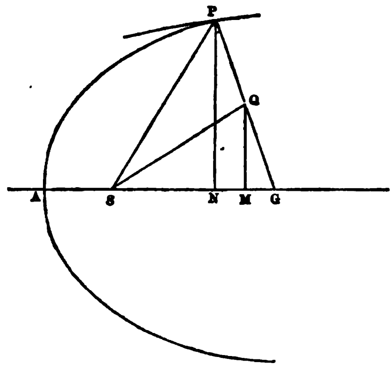
$$\{y^2 + (x - a)^2\}\{y\tan\beta - x\tan^2\beta - a\} = 0.$$

If $y^2 + (x - a)^2 = 0$, then $y = 0$, $x = a$, these are the co-ordinates of the focus, but *not* of a point in the locus of (1) and (2), for these values do not satisfy both (1) and (2), hence the required locus is

$$y = x\tan\beta + a\cot\beta,$$

which is the equation to a tangent inclined at an angle β to the axis.

33. Let $AM = x$, $MQ = y$;
$AN = x'$. $MQ = y'$.



Since $SG = SP$, the line SQ bisects PG;

$$\therefore\ QM = \frac{1}{2} PN, \text{ and } GM = \frac{1}{2} GN = NM.$$

$$\text{Hence } y = \frac{1}{2} y', \text{ and } x = AM = x' + a.$$

$$\text{Now } y'^2 = 4ax'; \quad \therefore\ 4y^2 = 4a(x - a);$$

$\therefore\ y^2 = a(x - a)$, the equation to a parabola whose latus rectum $= a$, and vertex is at S.

35. Let h be the abscissa of the given point from which the normals are drawn; then the equation to the normal is

$$y = -\frac{y}{2a}(x - h) \dots\dots\dots\dots\dots\dots (1),$$

and the equation to the parabola is

$$y^2 = 4ax \dots\dots\dots\dots\dots\dots\dots\dots (2);$$

therefore, eliminating the variable parameter a by means of (1) and (2), we have

$$y = -\frac{2x}{y}(x - h),$$

$$\text{or } y^2 + 2x(x - h) = 0,$$

for the equation to the locus which is an ellipse, whose minor axis (coincident with the axis of the parabola) is h, and major axis is $h\sqrt{2}$.

41. The equations to the parabola and circle referred to the tangents as axes are

$$\left(\frac{x}{h}\right)^{\frac{1}{2}} + \left(\frac{y}{k}\right)^{\frac{1}{2}} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\text{and } x + y - 2\,(xy)^{\frac{1}{2}} \sin\frac{1}{2}\,\omega = c\,.\cot\frac{1}{2}\,\omega \;\ldots\ldots\ldots\ldots (2).$$

Now for the points of intersection of the two curves, the values of x and y in equations (1) and (2) must be the same, therefore, eliminating y, we have

$$x\,\{h + k + 2\,(xh)^{\frac{1}{2}} \sin\frac{1}{2}\,\omega\} - 2\,\{k\,(h)^{\frac{1}{2}} + h\,(k)^{\frac{1}{2}} \sin\frac{1}{2}\,\omega\}$$

$$+ h\,\{k - c\cot\frac{1}{2}\,\omega\} = 0,$$

and since for the point of contact x has but one value, this quadratic must have its roots equal;

$$\therefore\ 4h\left(h + k + 2\sqrt{hk}\sin\frac{\omega}{2}\right)\left(k - c\cot\frac{\omega}{2}\right) = 4hk\left(k^{\frac{1}{2}} + h^{\frac{1}{2}}\sin\frac{\omega}{2}\right)^2;$$

$$\therefore\ \left\{h + k + 2\,(hk)^{\frac{1}{2}}\sin\frac{\omega}{2}\right\}\left(k - c\cot\frac{\omega}{2}\right)$$

$$= k\left\{k + 2\,(hk)^{\frac{1}{2}}\sin\frac{\omega}{2} + h\sin^2\frac{\omega}{2}\right\},$$

$$\text{whence } 2c = \frac{hk\sin\omega}{h + k + 2\,(hk)^{\frac{1}{2}}\sin\frac{1}{2}\,\omega}.$$

48. Let (x'', y'') be the co-ordinates of Q; (x', y') those of P; and therefore $\frac{a^2}{x'}$, $-\frac{4a^2}{y'}$ those of p, then the equations to PQ and Qp are

$$y - y' = \frac{y'' - y'}{x'' - x'}\,(x - x') \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\text{and } y - y'' = -\frac{x'}{y'}\left(\frac{4a^2 + y'y''}{a^2 - x'x''}\right)(x - x'') \ldots\ldots\ldots (2).$$

In both these equations let $x = -a$, then from (1) we have

$$y = -\frac{y'' - y'}{x'' - x'}\,(a + x) + y',$$

or since $y''^2 = 4ax''$ and $y'^2 = 4ax'$,

$$y = -\frac{4a^2 + y'^2}{y'' + y'} + y';$$

$$\therefore DR = \frac{y'y'' - 4a^2}{y' + y''},$$

and from (2) we have

$$y - y'' = \frac{x'}{y'}\left(\frac{4a^2 + y'y''}{a^2 - x'x''}\right)(a + x'');$$

$$\therefore y = -\frac{4a^2(y' + y'')}{y'y'' - 4a^2} = Dr;$$

$$\therefore DR \times Dr = 4a^2.$$

51. See solution of 33.

56. The focus of the parabola being the pole, we have

$$\frac{2a}{r} = 1 + \cos\theta \text{..................(1)},$$

$$\text{and } \frac{2a}{r'} = 1 - \cos\theta \text{..................(2)},$$

$$\text{from (1), } \left\{\left(\frac{2a}{r}\right) - 1\right\}^2 = \cos^2\theta \text{............(3)},$$

$$\text{from (2), } \left\{\left(\frac{2a}{r'}\right) - 1\right\}^2 = \sin^2\theta \text{............(4)},$$

$$\text{whence } \left(\frac{2a}{r} - 1\right)^2 + \left(\frac{2a}{r'} - 1\right)^2 = 1;$$

$$\therefore \left(\frac{1}{r} - \frac{1}{2a}\right)^2 + \left(\frac{1}{r'} - \frac{1}{2a}\right)^2 = \frac{1}{(2a)^2}.$$

64. Referring the parabola to the tangent QF, and the diameter at Q as axes, let (h, k) be the co-ordinates of the point R; then the equation to the line REF is

$$y = -k,$$

and when this line meets the parabola

$$k^2 = lx,$$

$$\text{whence } x = \frac{k^2}{l} = EF; \quad \therefore RE = h - \frac{k^2}{l};$$

$$\therefore EF : RE = \frac{k^2}{l} : h - \frac{k^2}{l} = k^2 : lh - k^2.$$

The equation to Qq is

$$y = -\frac{k}{h}.x \text{ . . . . } (1).$$

Eliminating x by means of (1), and the equation to the parabola $y^2 = lx$, we have

$$y = -\frac{lh}{k} = \text{the ordinate of } q;$$

$$\therefore RQ : Rq = k : \frac{lh}{k} - k = k^2 : lh - k^2;$$

$$\therefore RQ : Rq = EF : ER.$$

66. The equation to the given line shews that it passes through the focus, and is inclined at an angle of 45° to the axes.

Let PSp be the given line, and let tangents be applied at the extremities P, p; these tangents will intersect at a point T, in the directrix, and the line ST will be perpendicular to the given line;

$$\therefore SP = \frac{2a}{1 - \cos\theta} = \frac{2\sqrt{2}}{\sqrt{2} - 1}.a,$$

$$\text{and } Sp = \frac{2\sqrt{2}}{\sqrt{2} + 1}.a; \quad \therefore PSp = 8a,$$

$$\text{and } ST = (SP \times Sp)^{\frac{1}{2}} = 2\sqrt{2} \times a.$$

The area of the parabolic segment $= \frac{2}{3} \Delta PTp$;

$$\therefore \text{area of segment} = \frac{2}{3} ST \times \frac{Pp}{2} = \frac{16\sqrt{2}}{3} a^2.$$

68. In the triangle ABC, let $AB = c$, $BC = r$, and $\angle ABC = \theta$. Draw CD perpendicular to AB. Then

$$\tan A = \frac{BD}{AD} \tan\theta \text{ . . . . } (1),$$

$$\text{and } BD = r.\cos\theta; \quad \therefore AD = c - r\cos\theta;$$

$$\therefore \tan A = \frac{r \sin\theta}{c - r\cos\theta} = \frac{2}{\tan\frac{1}{2}\theta} \text{ by hypothesis};$$

$$\therefore \frac{r\sin\theta}{c - r\cos\theta} = \frac{2\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta};$$

$$\therefore r\sin^2\frac{1}{2}\theta = c - r\cos\theta;$$

$$\therefore r = \frac{c}{\sin^2\frac{1}{2}\theta + \cos\theta} = \frac{2c}{1+\cos\theta};$$

$\therefore$ the locus of C is a parabola.

Ellipse.

Ex. 4.

1. $$3\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + 2\left(y^2 + \frac{1}{2}y + \frac{1}{16}\right) = 1 + \frac{1}{3} + \frac{1}{8} = \frac{35}{24};$$

$$\therefore \frac{\left(x - \frac{1}{3}\right)^2}{\frac{35}{72}} + \frac{\left(y + \frac{1}{4}\right)^2}{\frac{35}{48}} = 1.$$

Hence the co-ordinates of the centre of ellipse, are

$$h = \frac{1}{3}, \quad k = -\frac{1}{4};$$

the semi-axes are $a = \left(\frac{35}{72}\right)^{\frac{1}{2}}, \quad b = \left(\frac{35}{48}\right)^{\frac{1}{2}}.$

3. To transform the equation, so that the rectangle of the co-ordinates may disappear, we must change the direction of the axes.

Let $x = t\cos\theta - u\sin\theta$; $y = t\sin\theta + u\cos\theta$.

Then

$$\begin{aligned}
5x &= 5t^2 \cos^2\theta - 10tu \cos\theta\sin\theta + 5u^2\sin^2\theta,\\
+2xy &= 2t^2\cos\theta\sin\theta + 2tu(\cos^2\theta - \sin^2\theta) - 2u^2\cos\theta\sin\theta,\\
+5y^2 &= 5t^2\sin^2\theta + 10tu\cos\theta\sin\theta + 5u^2\cos^2\theta,\\
-12x &= -12t\cos\theta + 12u\sin\theta,\\
-12y &= -12t\sin\theta - 12u\cos\theta;
\end{aligned}$$

$$\therefore\ 0 = t^2(5+\sin 2\theta) + 2tu\cos 2\theta + u^2(5-\sin 2\theta) - 12t(\cos\theta+\sin\theta) + 12u(\sin\theta - \cos\theta).$$

To θ, give such a value, that the coefficient of tu may $=0$;

$$\therefore \cos 2\theta = 0 = \cos 90^\circ; \quad \therefore \theta = 45^\circ,$$

and the equation becomes

$$6t^2 + 4u^2 - 12\sqrt{2}t = 0,$$

$$3(t^2 - 2\sqrt{2}t + 2) + 2u^2 = 6;$$

$$\therefore \frac{(t-\sqrt{2})^2}{2} + \frac{u^2}{3} = 1; \quad \therefore h = \sqrt{2},\ k = 0; \quad a = \sqrt{2},\ b = \sqrt{3}.$$

6. Transforming the origin to a point α, β, and equating the coefficient of x and y to zero, we find

$$\alpha = 1, \quad \beta = -1,$$

and the equation of the curve when the origin of co-ordinates is removed to the centre, becomes

$$3x'^2 + 2x'y' + y'^2 - 2 = 0.$$

Turning the axes through an angle θ, and equating to zero the coefficient of xy, we get

$$-6\sin\theta\cos\theta + 2(\cos^2\theta - \sin^2\theta) + 2\sin\theta\cos\theta = 0,$$

$$\text{whence } \tan 2\theta = 1, \text{ and } \therefore \theta = \frac{1}{8}\pi;$$

$$\therefore \sin^2\theta = \frac{\sqrt{2}-1}{2\sqrt{2}}, \quad \cos^2\theta = \frac{\sqrt{2}+1}{2\sqrt{2}},$$

and the equation to the curve then becomes

$$(2+\sqrt{2})x''^2 + (2-\sqrt{2})y''^2 = 2;$$

therefore the axes are $(2-\sqrt{2})^{\frac{1}{2}}$ and $(2+\sqrt{2})^{\frac{1}{2}}$,

$$\text{and } h = (\alpha^2+\beta^2)^{\frac{1}{2}} \cos 67^\circ\, 30' = \left(1 - \frac{1}{\sqrt{2}}\right)^{\frac{1}{2}};$$

$$k = (\alpha^2+\beta^2)^{\frac{1}{2}} \sin 67^\circ\, 30', \quad \text{or } k = \left(1 + \frac{1}{\sqrt{2}}\right)^{\frac{1}{2}}.$$

8. Transforming the origin to a point (α, β), we have, after equating to zero the coefficients of x and y,

$$\alpha = -2, \quad \beta = 0,$$

and the equation to the curve then becomes

$$16x'^2 + 7y'^2 + 16x'y' - 36 = 0 \ldots\ldots\ldots\ldots\ldots(1).$$

Now transfer the axis of y through an angle of 30°, and we have

$$y' = \frac{2y''}{\sqrt{3}}, \quad x' = x'' - \frac{y''}{\sqrt{3}},$$

which values being substituted in (1), we have for the transformed equation

$$16x''^2 + 4y''^2 = 36;$$

therefore the semi-axes are $\frac{3}{2}$ and 3.

11. The centre of the ellipse being the origin of rectangular co-ordinates, the equation to the circle will be

$$(x - ae)^2 + (y - r)^2 = r^2,$$

and when the circle meets the axis of y,

$$x = 0, \; y = b;$$

$$\therefore\ (ae)^2 + b^2 - 2br = 0,$$

$$2br = b^2 + a^2e^2 = a^2,$$

$$\text{or } b : a :: a : 2r.$$

13. Let $SP = h$, $PH = k$, $SH = l$; $2s = h + k + l$. Then

$$\tan\frac{PSH}{2}\tan\frac{PHS}{2} = \sqrt{\frac{(s-h)(s-l)}{s(s-k)}}\cdot\sqrt{\frac{(s-k)(s-l)}{s(s-h)}} = \frac{s-l}{s}.$$

$$\text{Now } s = \frac{1}{2}\{(a+ex)+(a-ex)+2ae\} = a(1+e),$$

$$s-l = a(1+e) - 2ae = a(1-e);$$

$$\therefore \tan\frac{PSH}{2}\tan\frac{PHS}{2} = \frac{1-e}{1+e}.$$

15. Using the same notation as in (13),

$$r = \left\{\frac{(s-h)(s-k)(s-l)}{s}\right\}^{\frac{1}{2}},$$

$$R = \frac{hkl}{4\{s(s-h)(s-k)(s-l)\}^{\frac{1}{2}}};$$

$$\therefore Rr = \frac{hkl}{4s} = \frac{2ae.hk}{4a(1+e)} \propto SP.HP.$$

18. Let x', y' be the co-ordinates of P; the equation to AQO will be

$$y = \frac{y'}{x'}(x+a) = m(x+a) \text{ suppose}\ldots\ldots\ldots\ldots(1).$$

To find the intersection of AQO with the ellipse, eliminate x from between this equation, and $a^2y^2 + b^2x^2 = a^2b^2$.

$$\text{Then } a^2y^2 + b^2\left(\frac{y}{m} - a\right)^2 = a^2b^2,$$

$$\left(a^2 + \frac{b^2}{m^2}\right)y^2 - \frac{2ab^2}{m}y = 0;$$

$$\therefore y = 0 \text{ and } y = \frac{2ab^2m}{m^2a^2+b^2} = \frac{2ab^2x'y'}{a^2y'^2+b^2x'^2} = \frac{2ab^2x'y'}{a^2b^2} = \frac{2x'y'}{a}.$$

$$\text{Hence } AQ : \frac{2x'y'}{a} = CP : y',$$

$$AO : a = CP : x';$$

$$\therefore AQ.AO = 2CP^2.$$

21. The equation to the tangent is, generally,

$$a^2yy' + b^2xx' = a^2b^2,$$

$$\text{and at } L,\quad x' = -ae, \text{ and } y' = \frac{b^2}{a};$$

$$\therefore\ a^2y\,.\frac{b^2}{a^2}-b^2x\,.\,ae=a^2b^2,$$

$$\text{or}\ \ y=a+ex.$$

If the ordinate y, or NR cut the ellipse in P, we have

$$SP=a+ex;$$

$$\therefore\ SP=NR.$$

26. Let h, k be the co-ordinates of the intersection of the tangents, then the equation to the chord joining the points of contact is

$$a^2ky+b^2hx=a^2b^2,$$

and since the chord passes through the focus, when $x=ae$, $y=0$;

$$\therefore\ b^2h\,.\,ae=a^2b^2;\quad \text{and}\ \therefore\ h=\frac{a}{e},$$

the equation to the directrix.

The equation to the perpendicular to the chord is

$$y=\frac{k}{h}\cdot\frac{a^2}{b^2}(x-ae),$$

$$\text{and when}\ \ x=\frac{a}{e},\quad y=\frac{ka}{he}=k;$$

therefore the perpendicular passes through the intersection of the tangents.

28. Let AY, $A'Z$ be the lengths of perpendiculars drawn from the extremities A, A' of the major axis upon the tangent at $P(x', y')$.

$$\text{Then}\ AY=AT\sin\theta=\left(\frac{a^2}{x'}+a\right)\sin\theta,$$

$$A'Z=A'T\sin\theta=\left(\frac{a^2}{x'}-a\right)\sin\theta;$$

$$\therefore\ AY.A'Z=\frac{a^2}{x'^2}(a^2-x'^2)\sin^2\theta=\frac{a^2}{x'^2}\cdot\frac{a^2y'^2}{b^2}\sin^2\theta=b^2\left(\frac{a^2y'}{b^2x'}\right)^2\sin^2\theta$$

$$=b^2\frac{\sin^2\theta}{\tan^2\theta}=b^2\cos^2\theta.$$

29. Let x, y be the co-ordinates of P; then

$$\tan\phi = \tan(APN + A'PN) = \frac{\dfrac{a+x}{y} + \dfrac{a-x}{y}}{1 - \dfrac{a^2-x^2}{y^2}} = \frac{2ay}{y^2 - (a^2 - x^2)}$$

$$= \frac{2ay}{y^2 - \dfrac{a^2}{b^2}y^2} = -\frac{2b^2}{ae^2y}.$$

If SY be the perpendicular from the focus S upon the tangent at P, then

$$\sin\theta = \frac{SY}{SP} = \frac{\left(ae + \dfrac{a^2}{x}\right)}{a + ex}\sin\psi = \frac{a}{x}\sin\psi, \text{ where } \tan\psi = -\frac{b^2x}{a^2y};$$

$$\therefore\ \sin\psi = \frac{-b^2x}{\sqrt{a^4y^2 + b^4x^2}};$$

$$\therefore\ \sin\theta = \frac{-ab^2}{\sqrt{a^4y^2 + b^4x^2}}.$$

$$\text{Hence } \tan\theta = -\frac{b^2}{aey} = \frac{e}{2}\tan\phi;$$

$$\therefore\ 2\tan\theta = e\tan\phi.$$

32. The equation to the tangent is

$$y = mx + (b^2 + a^2m^2)^{\frac{1}{2}}, \text{ or } (y - mx)^2 = b^2 + a^2m^2,$$

$$\text{or } (x^2 - a^2)m^2 - 2xy\,m + y^2 - b^2 = 0,$$

$$\text{where } m = \tan\theta.$$

If therefore x, y, be the co-ordinates of the intersection of the tangents, the two values of m in this quadratic will be $\tan\theta$, $\tan\theta'$. Hence we have for the equation to the required locus

$$(y^2 - b^2) = c(x^2 - a^2),$$

which, c being a negative quantity, represents an ellipse whose centre is the centre of the given ellipse.

33. Let O be the centre of the circle, $\angle PCT = \theta$, $\angle OCT = \beta$. Draw CD parallel to Tt, then $\angle ACD = \beta$.

$$\text{Now } PL = CL - CP = 2r\cos(\theta - \beta) - CP,\quad \frac{CT}{2r} = \cos\beta;$$

$$\therefore\ 2r = \frac{a^2}{CP\cos\theta\cos\beta},$$

$$\text{and } PL = \frac{a^2\cos(\theta-\beta)}{CP\cos\theta\cos\beta} - CP = \frac{a^2(\cos\theta + \sin\theta\tan\beta)}{CP\cos\theta} - CP$$

$$= \frac{a^2(1+\tan\theta\tan\beta)}{CP} - CP.$$

$$\text{But } \tan\theta\tan\beta = \frac{b^2}{a^2};$$

$$\therefore\ PL = \frac{a^2+b^2-CP^2}{CP} = \frac{CD^2}{CP}$$

= half the chord of curvature at P in the direction of C,

$$CL = \frac{CD^2}{CP} + CP = \frac{CD^2+CP^2}{CP} = \frac{a^2+b^2}{CP};$$

$$\therefore\ CL\,.\,CP = a^2+b^2, \text{ a constant quantity.}$$

38. Let Q be any other point in the curve; x and $x+h$ the abscissæ of P and Q respectively.

$$\text{Then } GQ^2 = \{(x+h)^2 + \frac{b^2}{a^2}a^2 - (x+h)^2\}$$

$$= e^2(x-h)^2 - 2e^2x(a+h) + e^4x^2 + b^2,$$

$$\text{and } GP^2 = \frac{b^2}{a^2}(a^2-e^2x^2) = (1-e^2)(a^2-e^2x^2)$$

$$= -e^2x^2 \qquad + e^4x^2 + b^2;$$

$$\therefore\ GQ^2 - GP^2 = e^2\{(x+h)^2 - 2x(x+h) + x^2\} = e^2h^2;$$

$\therefore$ GQ is always greater than GP.

42. Let x', y' be the point at which the tangent RZ touches the ellipse. Then we have for the elimination of x', y' the equations

$$a^2by' + b^2a\sqrt{3}\,.\,x' = a^2b^2,$$

$$\text{and } a^2y'^2 + b^2x'^2 = a^2b^2,$$

$$\text{whence } \left(\frac{a^2b^2 - b^2a\sqrt{3}\,.\,x'}{ab}\right)^2 + b^2x'^2 = a^2b^2,$$

$$\text{whence } x' = \frac{a\sqrt{3}}{2};$$

$\therefore$ the $\Delta\, BRZ$ is isosceles, or $BZ = RZ$.

43. Let (h, k) be the given point. The equation to the curve is

$$a^2y^2 + b^2x^2 = a^2b^2 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the equation to the chord is

$$y - k = m(x - h) \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

At the intersection of (1) and (2) there is

$$(a^2m^2 + b^2)x^2 - 2a^2m(mh - k)x + a^2(k^2 - 2hkm + m^2b^2 - b^2) = 0.$$

The abscissa x of the middle point of (1) and (2) will be equal to the semi-sum of the roots of this equation, hence

$$x = \frac{a^2m^2h - a^2mk}{a^2m^2 + b^2} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Also y being the ordinate of the middle point

$$y - k = m(x - h) \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Eliminating m between (3) and (4), we have for the equation to the required locus

$$\frac{y^2 - ky}{b^2} + \frac{x^2 - hx}{a^2} = 0,$$

$$\text{or } \frac{(2y - k)^2}{b^2} + \frac{(2x - h)^2}{a^2} = \frac{k^2}{b^2} + \frac{h^2}{a^2},$$

which is therefore an ellipse, the co-ordinates of whose centre are $\frac{h}{2}$ and $\frac{k}{2}$ and whose semi-axes are

$$\frac{1}{2b}(a^2k^2 + b^2h^2)^{\frac{1}{2}} \text{ and } \frac{1}{2a}(a^2k^2 + b^2h^2)^{\frac{1}{2}}.$$

46. Let (x', y') be the co-ordinates of P, and (x, y) those of the centre of the circle, O; then

$$\text{the area of } \Delta\, SPH = aey' = y(1 + e)a \ldots\ldots\ldots (1).$$

Again, since the line from P through O bisects the angle SPH, we have

$$\tan OHS = \frac{\sin PHS}{1 + \cos PHS},$$

$$\text{but } \sin PHS = \frac{y'}{a - ex'}, \text{ and } \cos PHS = \frac{ae - x'}{a - ex'},$$

$$\text{Also}\ \tan OHS = \frac{y}{ae - x},$$

$$\text{whence}\ \frac{y}{ae-x} = \frac{y'}{(1+e)(a-x')} \ldots\ldots\ldots\ldots (2).$$

Therefore from (1) and (2) we have

$$y' = \frac{1+e}{e}y; \quad x' = \frac{x}{e},$$

which values being substituted in the equation

$$(1-e^2)x'^2 + y'^2 = a^2(1-e^2),$$

$$\text{we have}\ (1+e)y^2 + (1-e)x^2 = a^2e^2(1-e)$$

for the required locus, which will be a concentric ellipse.

48. Let G be the foot of the normal at P, then

$$\frac{SG}{HG} = \frac{SP}{HP} = \frac{PY}{PZ} = \frac{SY}{HZ} = \frac{SQ}{QZ};$$

therefore Q the point of intersection of SZ, HY is in the normal at P; also

$$\frac{PQ}{HZ} = \frac{PY}{YZ} = \frac{SG}{SH} = \frac{QG}{HZ};$$

therefore PG is bisected in Q.

Hence, if (x', y') be the co-ordinates of P, and (x, y) those of Q, we have

$$x = OG + \frac{1}{2}GN = e^2x' + \frac{1}{2}(1-e^2)x' = \frac{1+e^2}{2}x',$$

$$y = \frac{PN}{2} = \frac{y'}{2};$$

$$\therefore\ x' = \frac{2x}{1+e^2},\quad y' = 2y;$$

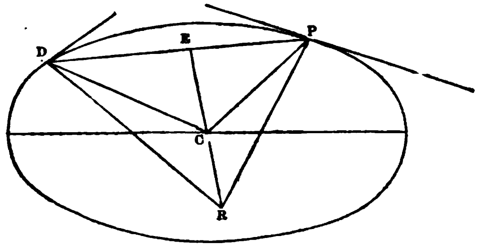
therefore substituting in the equation

$$a^2y'^2 + b^2x'^2 = a^2b^2,$$

$$\text{we have}\ \left\{\frac{2x}{a(1+e^2)}\right\}^2 + \frac{4y^2}{b^2} = 1$$

for the equation to the locus, which is an ellipse, whose major axis is $a(1+e^2)$, and minor is b, or $a(1-e^2)^{\frac{1}{2}}$.

50. Let $\bar{x}, \bar{y}$, be the co-ordinates of R,



$$y - y' = \frac{a^2 y'}{b^2 x'}(x - x') \text{ equation to } PR,$$

$$y - y'' = \frac{a^2 y''}{b^2 x''}(x - x'') \ldots\ldots\ldots\ldots\ldots DR;$$

$$\therefore \bar{y}\left(\frac{1}{y'} - \frac{1}{y''}\right) = \frac{a^2}{b^2}.\bar{x}\left(\frac{1}{x'} - \frac{1}{x''}\right);$$

$$\therefore \frac{\bar{y}}{\bar{x}} = \frac{a^2}{b^2}.\frac{y'y''}{x'x''}.\frac{x'' - x'}{y'' - y'} = -\frac{x'' - x'}{y'' - y'}.$$

$$\left.\begin{aligned} \text{Now } y'' &= \frac{b}{a}x' \\ x'' &= -\frac{a}{b}y' \end{aligned}\right\}; \quad \therefore \frac{y''}{x''} = -\frac{b^2}{a^2}.\frac{x'}{y'}; \quad \therefore \frac{a^2 y'y''}{b^2 x'x''} = -1.$$

Hence the line RCE is perpendicular to PD.

51. The equation to CP being

$$y = mx,$$

the equation to its conjugate CD will be

$$y = -\frac{b^2}{a^2 m}x,$$

when CP meets the directrix, we have

$$x = +\frac{a}{e}, \quad y = +\frac{ma}{e},$$

and when CD meets the directrix, we have

$$x = +\frac{a}{e}, \quad y = -\frac{b^2}{aem}.$$

Therefore for the perpendicular to CD, we have

$$y - \frac{ma}{e} = \frac{ma^2}{b^2}\left(x - \frac{a}{e}\right) \text{.................. (1)},$$

and for the perpendicular to CP, we have

$$y + \frac{b^2}{mae} = -\frac{1}{m}\left(x - \frac{a}{e}\right) \text{............... (2)},$$

If $y = 0$ in equations (1) and (2), we get

$$(1) \quad x = \frac{a}{e} - \frac{b^2}{ae} = \frac{a}{e}\left(1 - \frac{b^2}{a^2}\right) = ae,$$

$$(2) \quad x = \frac{a}{e} - \frac{b^2}{ae} = ae.$$

Hence, these perpendiculars intersect in the nearer focus.

53. Let Pp be any diameter of an ellipse, (x', y') the co-ordinates of P, and therefore $-x'$, $-y'$ those of p. Take R any point in the ellipse, and join RP, Rp, and let the co-ordinates of R be (x, y). Then if m, m' be the tangents of the angles which RP, Rp make with the axis of x respectively, we shall have

$$m = \frac{y - y'}{x - x'}, \quad m' = \frac{y + y'}{x + x'}, \text{ and } \therefore\ mm' = \frac{y^2 - y'^2}{x^2 - x'^2},$$

$$\text{also } a^2y^2 + b^2x^2 = a^2b^2, \text{ and } a^2y'^2 + b^2x'^2 = a^2b^2,$$

$$\text{whence } y^2 - y'^2 = -\frac{b^2}{a^2}(x^2 - x'^2);$$

$$\therefore\ mm' = -\frac{b^2}{a^2},$$

which shews that if pR be parallel to a diameter, RP will be parallel to the conjugate diameter.

59. EE' being parallel to the tangent at P, and the focal distances making equal angles with the tangent, the angles $EE'P$, $E'EP$ are equal;

$$\therefore\ PE' = PE = a, \text{ (vide Ex. 49)};$$

$$\therefore\ a - ex' + HE = a;$$

$$\therefore\ HE = ex' = SE.$$

Again, from Trigonometry, we have for the radius of the circle circumscribing $\Delta\, SEC$,

$$R = \frac{SE}{2 \sin C} \text{.................... (1)},$$

and for the circle circumscribing Δ *CHE*,

$$R = \frac{HE}{2 \sin C} \cdots\cdots\cdots\cdots\cdots\cdots\cdots (2).$$

Equations (1) and (2) show that the circles are equal.

60. $$a'^2 + b'^2 = a^2 + b^2,$$

$$\text{and } a'b' \sin\gamma = ab;$$

$$\therefore\ a' - b' = \left(a^2 + b^2 - \frac{2ab}{\sin\gamma}\right)^{\frac{1}{2}}.$$

Now $\sin\gamma$ is least, when $\dfrac{2ab}{\sin\gamma}$ is greatest, i.e. is $= a^2 + b^2$,

at which value $a' = b'$,

$$\sin\gamma = \frac{2ab}{a^2 + b^2}; \quad \therefore \frac{1 + \sin\gamma}{1 - \sin\gamma} = \frac{(a+b)^2}{(a-b)^2};$$

$$\therefore \frac{\cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}}{\cos\frac{\gamma}{2} - \sin\frac{\gamma}{2}} \text{ or } \frac{1 + \tan\frac{\gamma}{2}}{1 - \tan\frac{\gamma}{2}} = \frac{1 + \frac{b}{a}}{1 - \frac{b}{a}}; \quad \therefore \tan\frac{\gamma}{2} = \frac{5}{8}.$$

61. Let x', y' be the co-ordinates of P.

Then $a'^2 = x'^2 + y'^2$,

$$b'^2 = \frac{a^2}{b^2} y'^2 + \frac{b^2}{a^2} x'^2;$$

$$\therefore\ a'^2 - b'^2 = \left(1 - \frac{b^2}{a^2}\right) x'^2 - \left(\frac{a^2}{b^2} - 1\right) y'^2;$$

$$\therefore \frac{a'^2 - b'^2}{a^2 - b^2} = \frac{x'^2}{a^2} - \frac{y'^2}{b^2} = \frac{CN}{CT} - \frac{PN}{CT'} = \frac{CN}{CP} \cdot \frac{CP}{CT} - \frac{PN}{CP} \cdot \frac{CP}{CT'}$$

$$= \cos\alpha \frac{\cos\beta}{\sin(\alpha + 90^\circ - \beta)} - \sin\alpha \frac{\sin\beta}{\sin(\alpha + 90^\circ - \beta)}$$

$$= \frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)}.$$

66. The co-ordinates of P being x', y', and those of D, x'', y'',

$$\text{then } x'' = \pm \frac{a}{b} y', \text{ and } y'' = \pm \frac{bx'}{a}.$$

Now $\cos\alpha = \frac{x'}{a'}$, $\sin\alpha = \frac{y'}{a'}$; $\therefore\ a'^2 \sin 2\alpha = 2x'y'$,

$$\cos\beta = \frac{x''}{b'},\ \sin\beta = \frac{y''}{b'};\ \therefore\ b'^2 \sin 2\beta = 2x''y'';$$

$$\therefore\ a'^2 \sin 2\alpha + b'^2 \sin 2\beta = 2x'y' + 2\left(\mp \frac{a}{b} y'\right)\left(\pm \frac{bx'}{a}\right) = 0.$$

70. Let P, Q, R be the points of contact of the sides A_1A_2, A_2A_3, A_3A_1 of the circumscribing triangle, and let p, q, r be the lengths of the semi-diameters parallel to these sides, then

$$\frac{A_1P}{A_1Q} = \frac{p}{q},\quad \frac{A_2R}{A_2P} = \frac{r}{p},\quad \frac{A_3Q}{A_3R} = \frac{q}{r};$$

$$\therefore\ \frac{A_1P \,.\, A_2R \,.\, A_3Q}{A_1Q \,.\, A_2P \,.\, A_3R} = \frac{p \,.\, r \,.\, q}{q \,.\, p \,.\, r} = 1,$$

whence $A_1P \,.\, A_2R \,.\, A_3Q = A_1Q \,.\, A_2P \,.\, A_3R$.

75. By formula (12) we have

$$\frac{1}{r^2} = \frac{1 - e^2 \cos^2\theta}{b^2} = \frac{(1-e^2) + e^2 \sin^2\theta}{b^2},$$

$$\text{and}\ \frac{1}{r'^2} = \frac{1 - e^2 \cos^2\theta'}{b^2} = \frac{(1-e^2) + e^2 \sin^2\theta'}{b^2};$$

$$\therefore\ \frac{1}{r^2} - \frac{1}{r'^2} = \frac{e^2 (\sin^2\theta - \sin^2\theta')}{b^2}$$

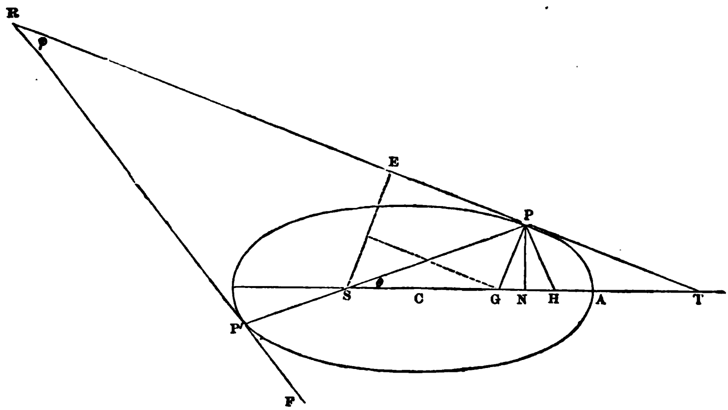
$$= \frac{a^2 - b^2}{a^2b^2} (\sin^2\theta - \sin^2\theta').$$

78. Let x', y' be the co-ordinates of P, SE a perpendicular from S on the tangent at P,

$$\tan SPE = \frac{SE}{EP} = \frac{ST \,.\, \sin T}{SG \,.\, \cos T} = \frac{\left(ae + \frac{a^2}{x'}\right) . \sin T}{(ae + e^2x') . \cos T} = \frac{a}{x'e} \cdot \frac{(a+ex')}{(a+ex')} \tan T$$

$$= \frac{a}{ex'} \cdot \frac{b^2x'}{a^2y'} = \frac{b^2}{aey'} = \frac{b^2}{aer \sin\theta} = \frac{a(1-e^2)}{e \sin\theta \dfrac{a(1-e^2)}{1 - e\cos\theta}}$$

$$= \frac{1 - e\cos\theta}{e \sin\theta}.$$



Hence $\tan SP'F = \dfrac{1 - e\cos(\pi+\theta)}{e\sin(\pi+\theta)} = \dfrac{1 + e\cos\theta}{-e\sin\theta}$

$$\tan\phi = \tan(SP'F - SPE)$$

$$= \frac{\dfrac{1+e\cos\theta}{-e\sin\theta} - \dfrac{1-e\cos\theta}{e\sin\theta}}{1 - \dfrac{1-e^2\cos^2\theta}{e^2\sin^2\theta}} = \frac{2e\sin\theta}{1-e^2}.$$

82. By the polar equations to the ellipse and circle, we have

$$AQ = \frac{2ab^2\cos\theta}{a^2-(a^2-b^2)\cos^2\theta}, \quad \text{and } AR = 2a\cos\theta,$$

$$A'Q' = \frac{2ab^2\sin\theta}{a^2-(a^2-b^2)\sin^2\theta}, \qquad A'R = 2a\sin\theta;$$

$$\therefore \frac{AR}{AQ} + \frac{A'R}{A'Q'} = \frac{a^2-(a^2-b^2)\cos^2\theta}{b^2} + \frac{a^2-(a^2-b^2)\sin^2\theta}{b^2} = \frac{a^2+b^2}{b^2}.$$

83. Let $SQ = p$, $\angle CSQ = \theta$,

then $\angle PSC = 2\theta$, and $SP = \dfrac{a(1-e^2)}{1-e\cos 2\theta}$;

$$\therefore \rho^2 = ae \times SP = \frac{a^2e(1-e^2)}{1-e\cos 2\theta};$$

or $\rho^2\{1 - e(\cos^2\theta - \sin^2\theta)\} = a^2e(1-e^2)$;

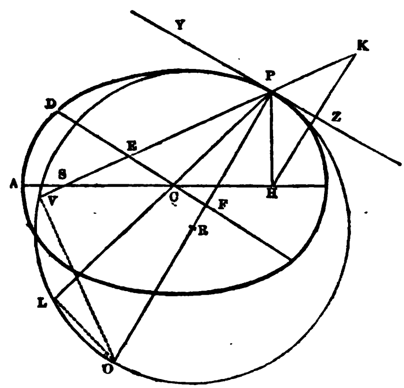
$$\therefore\ x^2 + y^2 - e(x^2 - y^2) = a^2e(1 - e^2),$$

$$\text{or}\ (1 - e)x^2 + (1 + e)y^2 = a^2e(1 - e^2),$$

$$\text{hence,}\ \frac{x^2}{a^2e(1+e)} + \frac{y^2}{a^2e(1-e)} = 1,$$

is the equation to an ellipse whose centre is S, and whose eccentricity is $\left(\frac{2e}{1+e}\right)^{\frac{1}{2}}$.

91. $$PL = PO \cos CPF = 2\rho \cdot \frac{PF}{CP}.$$



$$\text{Now}\ \rho = \frac{b'^3}{ab},\ \text{by formula (13),}$$

$$\text{and}\ \frac{PF}{CP} = \frac{ab}{a'b'},\ \text{from formula (9);}$$

$$\therefore\ PL = 2\frac{b'^3}{ab} \cdot \frac{ab}{a'b'} = 2\frac{b'^2}{a'} = \frac{2CD^2}{CP}.$$

$$\text{Also,}\ PV = PO \cos VPO = 2\rho\frac{PF}{PE}$$

$$= 2\frac{b'^3}{ab} \cdot \frac{ab}{ab'},\ \text{since}\ PE = AC = a,\ \text{(vid. 56),}$$

$$= \frac{2b'^2}{a} = \frac{2CD^2}{CA}.$$

92. Let (x, y) be the point of concourse of the tangents.

Then $y'^2 = b^2 - \frac{b^2}{a^2} x'^2$ is the equation to the ellipse (1),

and $y' = y + m(x' - x)$, ... tangent (2).

Eliminating y' we get after reduction, since the roots of the resulting quadratic in x' will be equal

$$(x^2 - a^2) m^2 - 2xym + (y^2 - b^2) = 0.$$

Now if m, m' be the roots of this equation,

$$m + m' = \frac{2xy}{x^2 - a^2}, \text{ and } mm' = \left(\frac{y^2 - b^2}{x^2 - a^2}\right),$$

$$(m - m') = t(1 + mm').$$

Hence, eliminating m and m', we have

$$t^2 \left(1 + \frac{y^2 - b^2}{x^2 - a^2}\right)^2 = \left(\frac{2xy}{x^2 - a^2}\right)^2 - 4\left(\frac{y^2 - b^2}{x^2 - a^2}\right);$$

$$\therefore t^2 (x^2 + y^2 - a^2 - b^2) = 4(a^2y^2 + b^2x^2 - a^2b^2),$$

which is the equation to the required locus.

HYPERBOLA.

Ex. 5.

1. $$y^2 - y + \frac{1}{4} - x^2 = \frac{1}{4};$$

$$\therefore \frac{\left(y - \frac{1}{2}\right)^2}{\frac{1}{4}} - \frac{x^2}{\frac{1}{4}} = 1.$$

Hence it appears that $h = 0$, $k = \frac{1}{2}$; $a = b = \frac{1}{2}$.

6. Let $x = t\cos\theta - u\sin\theta$, $y = t\sin\theta + u\cos\theta$.

Then $xy = t^2\cos\theta\sin\theta + tu(\cos^2\theta - \sin^2\theta) - u^2\cos\theta\sin\theta$,

$-x^2\tan\alpha = -t^2\cos^2\theta\tan\alpha + 2tu\cos\theta\sin\theta\tan\alpha - u^2\sin^2\theta\tan\alpha;$

$$\therefore \frac{1}{4} b^2 \cot\alpha = t^2\left(\frac{1}{2}\sin 2\theta - \cos^2\theta\tan\alpha\right) - u^2\left(\frac{1}{2}\sin 2\theta + \sin^2\theta\tan\alpha\right),$$

if $\cos 2\theta + \sin 2\theta \tan\alpha = 0$;

$$\text{and } \therefore \tan 2\theta = -\cot\alpha = \tan\left(\pm\frac{\pi}{2} + \alpha\right);$$

$$\therefore \theta = \pm\frac{\pi}{4} + \frac{\alpha}{2},$$

which values express the inclinations of the axes of t and u to that of x.

Now $\sin 2\theta = \sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha$, and $\cos 2\theta = -\sin\alpha$;

$$\therefore \sin 2\theta - (1 + \cos 2\theta)\tan\alpha = \cos\alpha + \frac{\sin^2\alpha}{\cos\alpha} - \tan\alpha = \frac{1 - \sin\alpha}{\cos\alpha},$$

$$\text{and } \sin 2\theta + (1 - \cos 2\theta)\tan\alpha = \cos\alpha + \frac{\sin^2\alpha}{\cos\alpha} + \tan\alpha = \frac{1 + \sin\alpha}{\cos\alpha};$$

$$\therefore u^2 \times \frac{1 + \sin\alpha}{\cos\alpha} - t^2 \times \frac{1 - \sin\alpha}{\cos\alpha} = \frac{b^2}{2}\cot\alpha.$$

Hence, $h = 0$, $k = 0$; and the ratio of the axes, or

$$\frac{a}{b} = \left(\frac{1 - \sin\alpha}{1 + \sin\alpha}\right)^{\frac{1}{2}} = \frac{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}} = \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right).$$

9. $$y^2 - 3xy + x^2 + 1 = 0.$$

The centre being the origin of co-ordinates;

$$h = 0, \quad k = 0.$$

In the given equation substitute for x and y as follows,

$$x = \frac{x' \sin(60 - \theta) + y' \sin(60 - \theta')}{\sin 60},$$

$$y = \frac{x' \sin\theta + y' \sin\theta}{\sin 60^\circ},$$

and equate to zero the coefficient of $x'y'$. Then, after reduction, we obtain

$$\tan 2\theta = \sqrt{3}; \quad \therefore \theta = 30^\circ,$$

and the reduced equation becomes

$$5y'^2 - \frac{1}{3}x'^2 = -1,$$

of which the semi-axes are

$$a = \sqrt{3}, \quad b = \frac{1}{\sqrt{5}}.$$

11. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and $\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1$, be the equations to the ellipse and hyperbola; then $CS^2 = a^2 - b^2 = a'^2 + b'^2$ since the curves have the same foci and centre; and at the points of intersection

$$\left(\frac{1}{a^2} - \frac{1}{a'^2}\right)x^2 + \left(\frac{1}{b^2} + \frac{1}{b'^2}\right)y^2 = 0;$$

$$\text{or } \frac{x^2}{y^2} = \frac{a^2a'^2}{b^2b'^2}\left(\frac{b^2 + b'^2}{a^2 - a'^2}\right) = \frac{a^2a'^2}{b^2b'^2}.$$

Now if θ, θ' be the angles which the tangents to the two curves at the point of intersection make with the axis of x,

$$\tan\theta = -\frac{b^2x}{a^2y}, \quad \tan\theta' = \frac{b'^2x}{a'^2y};$$

$$\therefore \tan\theta\tan\theta' = -\frac{b^2b'^2}{a^2a'^2}.\frac{x^2}{y^2} = -1;$$

or the tangents at the points of intersection of the two curves are at right angles to each other.

14. $$y^2 = \frac{b^3}{(a^2+b^2)^{\frac{1}{2}}}; \quad \therefore \frac{x^2}{a^2} = 1 + \frac{b}{(a^2+b^2)^{\frac{1}{2}}},$$

$$\text{or } x^2 = a^2 + \frac{a^2b}{(a^2+b^2)^{\frac{1}{2}}},$$

$$\text{hence } CP^2 = x^2 + y^2 = a^2 + b\,(a^2+b^2)^{\frac{1}{2}},$$

$$CD^2 = CP^2 - a^2 + b^2 = b^2 + b\,(a^2+b^2)^{\frac{1}{2}};$$

$$\therefore CY^2 = \frac{a^2b^2}{CD^2} = b\,(a^2+b^2)^{\frac{1}{2}} - b^2;$$

$$\text{hence } PY^2 = CP^2 - CY^2 = a^2 + b^2,$$

$$\text{or } PY = CS.$$

19. Let x, y, x', y' be the co-ordinates of P and D respectively; then the equation to the normal at P is

$$y_{,}=-\frac{a^2y}{b^2x}\cdot x_{,}+\left(1+\frac{a^2}{b^2}\right)y=-\frac{x'}{y'}\cdot x_{,}+\left(1+\frac{a^2}{b^2}\right)y \ldots\ldots (1).$$

Similarly the equation to the normal at D is

$$y_{,}=-\frac{x}{y}x_{,}+\left(1+\frac{a^2}{b^2}\right)y' \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

from (1) and (2) we obtain

$$y_{,}(y'-y)=-(x'-x)\,x_{,},$$

but the equation to a line through C perpendicular to PD is

$$Y=-\frac{x'-x}{y'-y}X;$$

$\therefore$ X, Y are co-ordinates of a point in this line; or the normals at P and D intersect in the perpendicular drawn from C upon PD.

21. Find the area included by the normals to a hyperbola which pass through the foci of the conjugate hyperbola.

Let S' be the focus of the conjugate hyperbola; $S'PK$ a normal; x, y the co-ordinates of P; then the equation to the normal $S'PK$ is

$$y'-y=-\frac{a^2y}{b^2x}(x'-x);$$

$$\text{and when } x'=0,\quad y'=\left(1+\frac{a^2}{b^2}\right)y=CS'=(a^2+b^2)^{\frac{1}{2}};$$

$$\therefore\ y=\frac{b^2}{(a^2+b^2)^{\frac{1}{2}}};$$

$$\text{also when } y'=0,\quad x'=CK=\left(1+\frac{b^2}{a^2}\right)x;$$

$$\text{and } \frac{x^2}{a^2}=1+\frac{y^2}{b^2}=1+\frac{b^2}{a^2+b^2}=\frac{a^2+2b^2}{a^2+b^2};$$

$$\therefore\ x'=\frac{\{(a^2+b^2)(a^2+2b^2)\}^{\frac{1}{2}}}{a},$$

and the area included between the four normals

$$=2\Delta\, CS'K=2x'y'=2\,\frac{(a^2+b^2)}{a}\,(a^2+2b^2)^{\frac{1}{2}}.$$

27. The equation to the hyperbola, and circle, referred to the asymptotes as axes, are

$$xy = \frac{a^2 + b^2}{4}, \text{ and } x + y - 2\sqrt{xy}\sin\frac{\theta}{2} = r\cot\frac{\theta}{2},$$

where θ = the angle between the asymptotes, and therefore $\tan\frac{\theta}{2} = \frac{b}{a}$.

Therefore for the points of intersection of the curves

$$x + \frac{a^2 + b^2}{4x} - (a^2 + b^2)^{\frac{1}{2}}\sin\frac{\theta}{2} = r\cot\frac{\theta}{2},$$

and in order that the circle may touch the hyperbola, the two values of x in this equation must be equal;

$$\therefore\ (a^2 + b^2)^{\frac{1}{2}}\sin\frac{\theta}{2} + r\cot\frac{\theta}{2} = (a^2 + b^2)^{\frac{1}{2}},$$

$$\text{whence } r = (a^2 + b^2)^{\frac{1}{2}}\left(1 - \sin\frac{\theta}{2}\right).\tan\frac{\theta}{2}$$

$$= (a^2 + b^2)^{\frac{1}{2}}\left\{\frac{(a^2 + b^2)^{\frac{1}{2}} - b}{(a^2 + b^2)^{\frac{1}{2}}}\right\}.\frac{b}{a}$$

$$= \frac{b}{a}\{(a^2 + b^2)^{\frac{1}{2}} - b\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now if the semi-latus rectum SL produced meet the asymptote in V, we have

$$SV = \frac{b}{a}(a^2 + b^2)^{\frac{1}{2}},$$

$$\text{but } SL = \frac{b^2}{a};$$

$$\therefore\ LV = \frac{b}{a}\{(a^2 + b^2)^{\frac{1}{2}} - b\} \ldots\ldots\ldots\ldots\ldots (2);$$

$$\therefore \text{ from (1) and (2) } r = LV.$$

30. Let x, y be the co-ordinates of the point of intersection of the ellipse and normal; the equation to the normal through the given point h, k, is

$$y - k = m(x - h) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the equation of the tangent at (x, y), is

$$y_{,} - y = -\frac{b^2 x}{a^2 y}(x_{,} - x) \dots\dots\dots\dots\dots\dots (2);$$

$$\therefore\ m = \frac{a^2 y}{b^2 x},$$

which being substituted in (1), we have

$$(y - k)\, b^2 x = (x - h)\, a^2 y,$$

for the equation to the locus, which is a rectangular hyperbola, passing through the given point, and having its asymptotes parallel to the axes of the ellipse.

32. The equation to the tangent of the hyperbola, in terms of its inclination to the major axis is,

$$y = mx + (a^2 m^2 - b^2)^{\frac{1}{2}} \dots\dots\dots\dots\dots\dots (1).$$

The equation to a perpendicular upon this tangent from the centre is

$$y = -\frac{1}{m} x \dots\dots\dots\dots\dots\dots (2).$$

At the intersection of (1) and (2) we have, by eliminating m between (1) and (2),

$$(x^2 + y^2)^2 = a^2 x^2 - b^2 y^2,$$

which is the equation to the required locus.

33. The equations of two right lines meeting the curve being

$$y + mx + n = 0 \dots\dots\dots\dots\dots\dots (1),$$

$$\text{and } y + m'x + n' = 0 \dots\dots\dots\dots\dots\dots (2).$$

Eliminating y by each of these equations and the equation to the hyperbola, finding the value of x in the resulting equation, and equating the radical in each with zero, we have

$$a^2 m^2 - b^2 - n^2 = 0, \text{ and } a^2 m'^2 - b^2 - n'^2 = 0.$$

The values of n and n' in (1) and (2) being substituted in these equations, and the equations arranged by the dimensions of m, m',

$$m^2 - \frac{2xy}{a^2 - x^2} m - \frac{b^2 + y^2}{a^2 - x^2} = 0, \text{ and } m'^2 - \frac{2xy}{a^2 - x^2} m' - \frac{b^2 + y^2}{a^2 - x^2} = 0.$$

The values of m, m' being the roots of either of these equations, let $mm' = -1$, then

$$\frac{b^2 + y^2}{a^2 - x^2} = 1, \quad \text{or } y^2 + x^2 = a^2 - b^2,$$

is the equation to the required locus, which is a circle whose radius $= (a^2 - b^2)^{\frac{1}{2}}$.

Sections of the Cone and General Problems.

Ex. 6.

1. Comparing the equation

$$y^2 = \frac{2d \sin \alpha \sin \theta}{\cos \alpha} \cdot x - \frac{\sin \theta \sin (2\alpha + \theta)}{\cos^2 \alpha} \cdot x^2,$$

$$\text{with } y^2 = \frac{2b^2}{a} x \mp \frac{b^2}{a^2} x^2,$$

$$\text{we have, } \frac{b^2}{a^2} = \pm \frac{\sin \theta \sin (2\alpha + \theta)}{\cos^2 \alpha} \quad \ldots\ldots\ldots\ldots\ldots (1),$$

the upper sign being used when the section is an ellipse, and the lower, when it is a hyperbola.

Substituting $\frac{1}{2}\pi$ for 2α in (1), we have

$$\frac{b^2}{a^2} = \pm 2 \sin \theta \cos \theta = \pm \frac{1}{2} \text{ by hypothesis;}$$

$$\therefore 2\theta = 150^\circ \text{ or } 30^\circ,$$

$$\text{and } \theta = 75^\circ \text{ or } 15^\circ;$$

$\therefore$ for the ellipse the inclination is

$$\pi - (45^\circ + 75^\circ) = 60^\circ,$$

and for the hyperbola

$$180^\circ - (135^\circ + 15^\circ) = 30^\circ.$$

3. From Ex. (1), we have

$$2b : 2a = \sqrt{\sin\theta \sin(2a+\theta)} : \cos a \text{............} (1),$$

$$\text{and } \theta = 180^\circ - (a+\delta);$$

$$\therefore \sin\theta = \sin(\delta + a),$$

$$\text{and } \sin(2a+\theta) = \sin\{180^\circ - (\delta - a) = \sin(\delta - a)\};$$

$\therefore$ substituting in (1), we have

$$\text{minor axis : major axis} = \{\sin(\delta+a)\sin(\delta-a)\}^{\frac{1}{2}} : \cos a.$$

5.
$$a^2 = h^2 + k^2 - 2hk(\cos^2 a - \sin^2 a)$$
$$= h^2 + k^2 - 2hk + 4hk \sin^2 a$$
$$= (h-k)^2 + 4hk \sin^2 a \text{.......................} (1).$$

From the extremities of the axis-major let fall the perpendiculars AF, $A'G$ upon the axis of the cone, and through C the middle point of AA' draw a plane parallel to the base of the cone, cutting the plane of the ellipse in BB', which is its minor axis, and the cone in the circle MBQ; then

$$BC^2 = MC \times CQ = A'G \times AF \text{..................} (2),$$

$$\text{but } A'G = k \sin a,$$

$$\text{and } AF = h \sin a,$$

which being substituted in (2), we have

$$\frac{b^2}{4} = hk \sin^2 a, \text{ or } b^2 = 4hk \sin^2 a \text{ ....} (3),$$

and from (1) and (3) we get

$$a^2 - b^2 = (h-k)^2.$$

7. The general equation is

$$y^2 = \frac{2d \sin a \sin\theta}{\cos a} . x - \frac{\sin\theta \sin(2a+\theta)}{\cos^2 a} . x^2 \text{.........} (1),$$

but for the parabola, $(2a+\theta) = \pi$;

$$\therefore \sin(2a+\theta) = 0, \text{ and } \sin\theta = \sin 2a;$$

therefore when the section is a parabola,

$$y^2 = \frac{2d \sin\theta \sin a}{\cos a} x$$
$$= 4d \sin^2 a . x;$$

$$\therefore \text{ the latus rectum} = 4d \sin^2 \alpha \ldots\ldots\ldots\ldots\ldots (1),$$

and when the plane passing through the directrix of the section and the vertex of the cone is perpendicular to the section, the expression for the latus rectum is $4d \cos 2\alpha$;

$$\therefore 2 \cos 2\alpha = 2 \sin^2 \alpha = 1 - \cos 2\alpha;$$

$$\therefore \cos 2\alpha = \frac{1}{3}.$$

13. Let the perpendicular from G, the foot of the normal meet the focal distance SP in L, and let the angle $PSG = \theta$.

The conic section being referred to its principal axis and directrix as co-ordinate axes, and (x', y') being the co-ordinates of P, we have for the abscissæ of S and G,

$$ep, \text{ and } ep(1 + e) + e^2x' \text{ respectively,}$$

and for the length of SP,

$$SP = e(p + x').$$

$$\begin{aligned} \text{Now, } PL &= SP - SL \\ &= SP - e^2(p + x') \cos\theta \\ &= SP - e\, SP \cos\theta \\ &= SP(1 - e\cos\theta) \\ &= \text{semi-latus rectum.} \end{aligned}$$

16. Let (x', y') be the co-ordinates of P, then the equation to the normal is

$$y - y' = -\frac{y'}{m + nx'}(x - x').$$

At the point G, $y = 0$;

$$\therefore AG = m + (n + 1)x' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equation to PK, perpendicular to the line $y = \dfrac{y'}{x'}x$, is

$$y - y' = -\frac{x'}{y'}(x - x').$$

At the point K, $y = 0$;

$$\therefore AK = \frac{y'^2 + x'^2}{x'} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2) and the equation to the curve, $y^2 = 2mx + nx^2$, we have

$$GK = \frac{2mx'}{x'} - m = m = \text{semi-latus rectum}.$$

18. Let PP', QQ' be any two chords which moving parallel to themselves intersect in O. Let O be the origin of co-ordinates, and the chords the co-ordinate axes; then the equation to the curve will be of the form

$$ay^2 + bxy + cx^2 + dy + ex + f = 0.$$

Let $x = 0$, then

$$ay^2 + dy + f = 0,$$

and the product of the roots being $\frac{f}{a}$, we have

$$QO \times Q'O = \frac{f}{a}.$$

$$\text{Similarly,} \quad PO \times P'O = \frac{f}{c};$$

$$\therefore\ QO.Q'O : PO.P'O = \frac{f}{a} : \frac{f}{c} = c : a,$$

and is therefore invariable.

21. The polar equation to any conic section, the focus being the pole, is

$$SP = \frac{SL}{1 + e\cos\theta},$$

also SP being produced to meet the curve in p, we have

$$Sp = \frac{SL}{1 + e\cos(\pi + \theta)} = \frac{SL}{1 - e\cos\theta};$$

$$\therefore \frac{1}{SP} + \frac{1}{Sp} = \frac{2}{SL},$$

which proves the proposition.

22. Refer the curve to the tangent as axes, its equation will be of the form

$$\left(\frac{x}{h} + \frac{y}{k} - 1\right)^2 + \mu xy = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let the equation to a straight line drawn through the origin be

$$\frac{x}{l}=\frac{y}{n}=r \text{ ....} (2).$$

Thus the distances from the origin of the points of intersection of (1) and (2) will be the values of r found from the equation

$$\left(\frac{lr}{h}+\frac{mr}{k}-1\right)^2+\mu lnr^2=0,$$

$$\text{or } \left(\frac{l}{h}+\frac{m}{k}-\frac{1}{r}\right)^2+\mu ln=0 \text{ ....} (3).$$

If r' and r'' be the roots of this equation, we have

$$\frac{1}{r'}+\frac{1}{r''}=2\left(\frac{l}{h}+\frac{m}{k}\right).$$

Also the equation to the chord of contact is

$$\frac{x}{h}+\frac{y}{k}-1=0 \text{ ....} (4).$$

Hence for the distance ($r_{,}$) of the point of intersection of (2) and (1), we have

$$\frac{lr_{,}}{h}+\frac{nr_{,}}{k}=1, \text{ or } \frac{1}{r_{,}}=\frac{l}{h}+\frac{n}{k} \text{ ....} (5).$$

From (4) and (5) we have

$$\frac{2}{r_{,}}=\frac{1}{r'}+\frac{1}{r''},$$

which proves the proposition.

25. (1) Let the point of bisection C, of the line SH, be taken as origin of rectangular co-ordinates, SH being the axis of x.

Assume $SC=b$, and $SP^2+HP^2=c^2$.

$$\text{Then, } SP^2=y^2+(x+b)^2,$$

$$HP^2=y^2+(x-b)^2;$$

$$\therefore\ c^2=2(y^2+x^2+b^2),$$

$$\text{whence } y^2+x^2=\frac{c^2}{2}-b^2$$

is the equation to the required locus, which is a circle, whose centre is C, and rad. $\left(\frac{c^2}{2}-b^2\right)^{\frac{1}{2}}$.

(2) In this case assume the point S as the origin of rectangular co-ordinates, and let (x, y) be the co-ordinates of P.

Let $SP=nc$, and $HP=c$, $SH=b$.

$$\text{Now } SP^2=y^2+x^2=n^2c^2 \quad\text{..................}\quad (1),$$

$$\text{and } c^2=y^2+(b-x)^2 \quad\text{..................}\quad (2),$$

and eliminating c between (1) and (2), we have for the required locus

$$y^2+x^2+\frac{2bn^2}{1-n^2}\cdot x-\frac{b^2n^2}{1-n^2}=0,$$

which is the equation of a circle, the co-ordinates of whose centre are

$$y_{\prime}=0,\quad x_{\prime}=\frac{bn^2}{n^2-1}.$$

28. Let S be any point in the circumference of a circle, and P the point of contact of any tangent to the circle of which C is the centre. Let Y be the extremity of the perpendicular from S on the tangent at P. Join CP and draw SN perpendicular to PC produced.

$$\text{Assume } SP=r,\ SY=\rho,\ CP=c.$$

$$\text{Then } SY=PN.$$

$$\text{And } SP^2=SC^2+CP^2+2CP\times CN;$$

$$\therefore r^2=2c^2+2c(\rho-c)=2c\rho;$$

$$\therefore \rho=\frac{r^2}{2c}$$

is the equation required.

Ex. 7.

2. The formulæ for transformation are

$$x=\frac{x'-y'}{\sqrt{2}},\quad y=\frac{x'+y'}{\sqrt{2}},$$

which being substituted in the proposed equation, we have

$$\begin{aligned}4x'^2y'^2&=a\cdot\frac{x'-y'}{\sqrt{2}}\cdot\left\{\frac{4(x'^2+y'^2)}{2}+2x'y'\right\}\\&=2^{\frac{1}{2}}\cdot a(x'-y')\{x'^2+y'^2+x'y'\}\\&=2^{\frac{1}{2}}\cdot a(x'^3-y'^3).\end{aligned}$$

LOCI.

Ex. 8.

3. Let AB be the given base, through D the middle of which draw the perpendicular DY; then taking DBX and DY for axes, and denoting the vertex of the triangle by (x, y), half the base by a, and the difference of the tangents by m, we have

$$m = \frac{y}{a-x} - \frac{y}{a+x} = \frac{2xy}{a^2-x^2}.$$

Hence the equation of the locus is

$$2xy = m(a^2 - x^2),$$

the equation to a hyperbola.

4. Let the base be taken as axis of x, and a perpendicular through the extremity A as axis of y; (y, x) the co-ordinates of the vertex, and c the length of the base AB, then if one base angle be α, and the other 2α, we have

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} \text{.....................} (1),$$

but $\tan\alpha = \dfrac{y}{x}$, and $\tan 2\alpha = \dfrac{y}{c-x}$.

Substituting these values in (1), we have

$$\frac{1}{c-x} = \frac{2x}{x^2-y^2};$$

therefore the equation of the locus sought is

$$y^2 = 3x^2 - 2cx,$$

which belongs to a hyperbola whose tranverse axis is two-thirds of the base.

6. Let C represent the base AB of the triangle, then, the altitude being constant, the locus of the vertex, C, is a parallel to AB. Hence, taking AB, AY for the rectangular axes, and putting x', y' for any point in the locus of C, we have for the equation of BC

$$y = \frac{y'}{x'-c}(x-c);$$

therefore the equation to the perpendicular to BC at its middle point is

$$y - \frac{y'}{2} = \frac{c - x'}{y'} \left(x - \frac{c + x'}{2} \right) \dots\dots\dots\dots \text{(1)}.$$

Similarly the equation to the perpendicular to AC is

$$y - \frac{y'}{2} = -\frac{x'}{y'} \left(x - \frac{x'}{2} \right) \dots\dots\dots\dots\dots\dots \text{(2)};$$

therefore for the intersection of (1) and (2),

$$x' \left(\frac{x'}{2} - x \right) = (c - x') \left(x - \frac{c + x'}{2} \right),$$

$$\text{whence } cx = \frac{c^2}{2};$$

$$\therefore x = \frac{c}{2}$$

is the equation to the locus, which is a straight line perpendicular to and bisecting the base.

9. Let AB be the given base, CD the given altitude, O the centre of the inscribed circle,

$$\text{let } CD = c, \; AB = 2a.$$

Assuming the middle point of AB as the origin of rectangular co-ordinates let (x, y) be the co-ordinates of O; and x' the abscissa of C; then

$$\tan A = \frac{c}{a + x'}, \quad \tan \frac{A}{2} = \frac{y}{a + x}.$$

$$\text{Also } \tan B = \frac{c}{a - x'}, \quad \tan \frac{B}{2} = \frac{y}{a - x};$$

$$\therefore \frac{c}{a + x'} = \frac{2y(a + x)}{(a + x)^2 - y^2} \dots\dots\dots\dots\dots\dots \text{(1)},$$

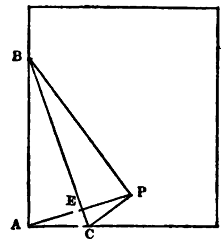
$$\text{and } \frac{c}{a - x'} = \frac{2y(a - x)}{(a - x)^2 - y^2} \dots\dots\dots\dots\dots\dots \text{(2)}.$$

Eliminating x' between (1) and (2), we have

$$\frac{2}{c} = \frac{1}{y} - \frac{y}{a^2 - x^2},$$

$$\text{or } c(a^2 - x^2 - y^2) = 2y(a^2 - x^2),$$

for the expression for the locus.



12. Let BCP be the position of the turned page.

Assume $AP=r$, $\angle PAC=\theta$, then since the triangles ABE, PBE are equal, we have $AE=\frac{r}{2}$, and the angle AEB a right angle;

$$\therefore\ AE=AC.\cos\theta,$$

$$\text{and } AE=AB.\cos\left(\frac{\pi}{2}-\theta\right)=AB.\sin\theta;$$

$$\therefore\ \frac{r^2}{4}=\frac{a^2}{2}\sin\theta.\cos\theta,$$

$$\text{or } r^2=a^2\sin 2\theta,$$

is the equation to the locus, which is a lemniscata.

13. Let $CN=x$, $NP=y$, be the co-ordinates of the point P, and let $CS=c$, then

$$SP=(PN^2+SN^2)^{\frac{1}{2}}=\{y^2+(c+x)^2\}^{\frac{1}{2}},$$

$$HP=(PN^2+HN^2)^{\frac{1}{2}}=\{y^2+(c-x)^2\}^{\frac{1}{2}};$$

therefore, by the hypothesis, we have

$$\{y^2+(c+x)^2\}\times\{y^2+(c-x)^2\}^{\frac{1}{2}}=c^2;$$

$$\therefore\ (y^2+c^2+x^2)^2-4c^2x^2=c^4;$$

$$\therefore\ y^4+2x^2y^2+x^4=2c^2x^2-2c^2y^2;$$

$$\therefore\ (y^2+x^2)^2=2c^2(x^2-y^2)$$

is the equation to the locus, which is the lemniscata of Bernouilli.

16. Let the two given lines be taken as axes of co-ordinates; and let m and n be the intercepts of the moveable line on these axes, then

$$4c^2=m^2+n^2-2mn\cos 2\alpha.$$

Now, let (x, y) be the co-ordinates of the middle point of the moveable line, then

$$x=\frac{m}{2},\quad y=\frac{n}{2},$$

$$\text{hence }\ c^2=x^2+y^2-2xy\cos 2\alpha$$

is the equation to the required locus, which is an ellipse.

Now change the direction of the axes by means of the formulæ

$$\begin{cases} x = \dfrac{x' \sin(2\alpha - \theta) - y' \cos(2\alpha - \theta)}{\sin 2\alpha} \\ y = \dfrac{x' \sin\theta + y' \cos\theta}{\sin 2\alpha}, \end{cases}$$

and equate the term involving x', y' to zero, and we have, after reduction,

$$\sin\theta = \sin(2\alpha - \theta)\,; \quad \therefore\ \theta = \alpha \ldots\ldots\ldots\ldots (2),$$

hence the equation to the curve referred to its axes is

$$x^2 \sin^4 \alpha + y^2 \cos^4 \alpha = c^2 \sin^2 \alpha \cos^2 \alpha\,;$$

$\therefore$ the semi-axes are $c \tan\alpha$, $c \cot\alpha$.

Equation (2) shews that the direction of one of the axes of the curve bisects the angle included between the given lines.

18. Assuming the two straight lines as co-ordinate axes, let (x, y) be the co-ordinates of the vertex; the equation to the parabola will then be

$$\left(\frac{x}{h}\right)^{\frac{1}{2}} + \left(\frac{y}{k}\right)^{\frac{1}{2}} = 1,$$

and it is easily found that

$$x = \frac{hk^4}{(h^2 + k^2)^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$y = \frac{h^3 k}{(h^2 + k^2)^2}, \text{ and } a = \frac{h^2 k^2}{(h^2 + k^2)^{\frac{3}{2}}},$$

$$\text{whence } \frac{k^3}{x} = \frac{h^3}{y} = \frac{h^{\frac{5}{4}} k^2}{a x^{\frac{1}{4}}} = \frac{h^2 k^{\frac{5}{4}}}{a y^{\frac{1}{4}}};$$

$$\therefore\ hk = \frac{a^8}{x^3 y^3};$$

$$\therefore\ \frac{k}{x^{\frac{1}{3}}} = \frac{a^8}{k x^3 y^3} \div y^{\frac{1}{3}} = \frac{a^8}{k x^3 y^{\frac{10}{3}}},$$

$$\text{whence } k = \frac{a^4}{x^{\frac{4}{3}} y^{\frac{5}{3}}}; \text{ and } \therefore\ h = \frac{a^4}{x^{\frac{5}{3}} y^{\frac{5}{3}}},$$

and substituting in (1) we have

$$x^{\frac{1}{2}} = \left\{\frac{a^2}{x^{\frac{5}{6}}y^{\frac{2}{3}}} \times \frac{a^8}{x^{\frac{8}{3}}y^{\frac{10}{3}}}\right\} \div \left\{\frac{a^8}{x^{\frac{10}{3}}y^{\frac{8}{3}}} + \frac{a^8}{x^{\frac{8}{3}}y^{\frac{10}{3}}}\right\}$$

$$= \frac{a^2}{x^{\frac{5}{6}}y^{\frac{2}{3}} + x^{\frac{1}{6}}y^{\frac{4}{3}}};$$

$$\therefore\ x^{\frac{4}{3}}y^{\frac{2}{3}} + x^{\frac{2}{3}}y^{\frac{4}{3}} = a^2$$

is the expression for the locus.

19. The equations to the parabola, and one of the tangents, are

$$y'^2 = 4ax' \dots\dots\dots\dots(1),$$

$$\text{and } y' - y = m(x' - x) \dots\dots\dots\dots(2),$$

where (x, y) are the co-ordinates of the intersection of the two tangents.

Eliminating y' between (1) and (2), we shall obtain a quadratic in x', the two roots of which must be equal: this condition will, after performing the requisite reductions, give us the equation

$$m^2x - my + a = 0.$$

Representing the two roots of this equation by m, m', we shall have

$$m + m' = \frac{y}{x}, \qquad \text{and} \quad 4mm' = \frac{4a}{x};$$

$$\therefore\ (m - m')^2 = \frac{y^2}{x^2} - \frac{4a}{x}, \quad 1 + mm' = \frac{x + a}{x}.$$

But by the hypothesis

$$(m - m') = \tan\beta\,(1 + mm').$$

Hence eliminating m and m', we have

$$y^2 - x^2\tan^2\beta - 2ax(2 + \tan^2\beta) - a^2\tan^2\beta = 0$$

for the equation to the required locus, which is a hyperbola of which a principal diameter coincides with the axes of x. The co-ordinates of the centre are

$$y = 0, \quad x = -a(2\cot^2\beta + 1).$$

The origin being removed to the centre, the equation becomes

$$y^2 - \tan^2\beta \,.\, x^2 = -4a^2 \operatorname{cosec}^2\beta.$$

$$\text{Now } \tan\phi = \frac{b}{a} = \tan\beta.$$

21. Let the centre of the ellipse C be the origin of rectangular co-ordinates, (x, y) the co-ordinates of P, and (x', y') those of Q; then the equation to AQ is

$$y = mx + c,$$

$$\text{and at } A \text{ we have } \quad c = ma;$$

$$\therefore\ y = m(x + a),$$

$$\text{and at } Q,\ y' = m(x' + a);\quad \therefore\ m = \frac{y'}{x' + a};$$

hence the equation to AQ is

$$y = \frac{y'}{x' + a}(x + a) \dots\dots\dots\dots (1),$$

and similarly that to $A'Q'$ is

$$y = -\frac{y'}{x' - a}(x - a) \dots\dots\dots\dots (2).$$

Eliminating x' and y' between (1), (2), and the equation to the ellipse, we get

$$x' = \frac{a^2}{x} \text{ and } y' = \frac{ay}{x}.$$

Substituting these values in the equation

$$a^2y'^2 + b^2x'^2 = a^2b^2,$$

we have for the required locus

$$a^2y^2 - b^2x^2 = -a^2b^2,$$

which is the equation to an hyperbola, whose centre is C, and transverse axis $2a$.

22. Let AX be the given line, $Q\ (h, k)$ the given point, T the point of contact of the circle and given line, (x, y) the

co-ordinates of the centre of the circle, and (x_1, y_1) those of any point on the circumference; then the equation to the circle is

$$(y_1 - y)^2 + (x_1 - x)^2 = r^2 \dots\dots\dots\dots\dots\dots (1),$$

but, passing through Q, it becomes

$$(k - y)^2 + (h - x)^2 = r^2 \dots\dots\dots\dots\dots\dots (2),$$

and being tangential to AX, we have

$$r = y \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3);$$

∴ from (2) and (3) we have

$$(k - y)^2 + (h - x)^2 = y^2,$$

$$(x - h)^2 = 2k\left(y - \frac{k}{2}\right),$$

for the equation to the locus, which is a parabola. Transferring the origin to the point (h, o) the equation becomes

$$x^2 + k = 2ky.$$

Ex. 9.

1. For $\sin\theta$ put $\frac{y}{r}$, and for $\cos\theta$, $\frac{x}{r}$, and the equation to the curve referred to rectangular co-ordinates becomes

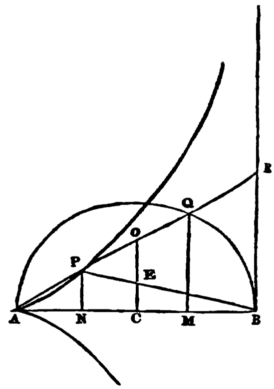
$$y^2 = \frac{x^3}{2a - x}; \quad \therefore y = \pm\left(\frac{x^3}{2a - x}\right)^{\frac{1}{2}}$$

The following table gives the corresponding values of x and y:

	1	2	3	4	5	6
values of x	o	a	$< 2a$	$2a$	$> 2a$	—
values of y	o	a	possible.	∞	impossible.	impossible.

Hence it appears from (1) that the curve passes through the origin, from (2) it bisects the semicircular arc AQB, from (3) there are possible values of y for all values of x less than $2a$, from (4) there is an infinite ordinate at B, or BR is an asymptote to the curve: from these values we thus obtain an infinite arc proceeding from A to meet the asymptote BR.

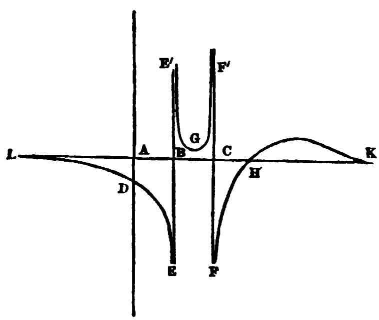
Again, from (5) for any value of x greater than $2a$, y is impossible, or no part of the curve is found to the right of the asymptote, and from (6) no part of the curve is on the left of A.



Also for every value of x there are two of y, equal and opposite, hence there is a branch below AB similar to the one above it.

The curve is the cissoid of Diocles, the equation to which has been found in Ex. 23.

6. When $x = 0$, $y = -\frac{3}{2}$, and the curve cuts the axis of y at D; as long as $x < 1$, y is negative, and becomes infinite when $x = 1$; therefore the ordinate BE, corresponding to $x = 1$, is an asymptote, and we thus get the branch DE.



When x is between 1 and 2, y' is positive, and becomes very great both when x is a little less than 2 and a little greater than 1, and gives the portion $E'GF'$.

When $x = 2$, y is infinite, so that the ordinate FF' is an asymptote.

When x lies between 2 and 3, y is negative, and gives the portion FH.

When $x = 3$, $y = 0$, and the curve cuts the axis at H. When $x > 3$, y is positive, and gives the portion HK; and when x is very great,

$$y = \frac{x}{x^2 - 3x} = \frac{1}{x-3} = 0,$$

therefore the axis of x is an asymptote.

When x is negative, the equation is

$$y = -\frac{x+3}{(x+1)(x+2)};$$

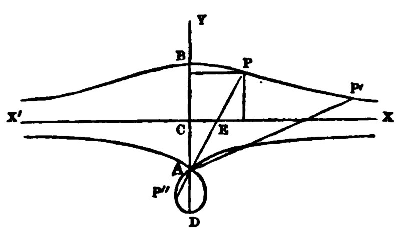
therefore y is always negative, and diminishes as x increases, and becomes 0 when $x = \infty$ and gives the branch DL.

(Hymers' *Conic Sections.*)

9. When $y = 0$, $x = \infty$, therefore the axis of x is an asymptote.

When $y = b$, $x = 0$, therefore the curve passes through B.

When $y < b$, x is possible, and when $y > b$, x is impossible; therefore the curve extends from the asymptote upwards to B, and no higher, and thus the branch BPP'.



Again, when $y = -a$, $x = 0$, and when $y = -b$, $x = 0$; therefore the curve passes through A and D if $CD = b$.

When $y < -a$, x is possible, hence there is a branch AX extending from A to the asymptote.

When $y > -a$, or $< -b$, x is possible; therefore the curve exists between A and D; the double value of x gives the same result along CX'.

MECHANICS

AND

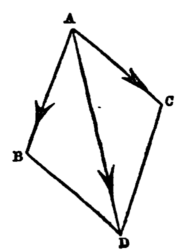
HYDROSTATICS.

STATICS.

Forces Acting in the same Plane.

Ex. 1.

1. Let AB, AC represent the two forces in direction and magnitude; then will the diagonal AD of the parallelogram $ABCD$ represent the resultant both in magnitude and direction.



Let $x =$ the resultant; then

$$AD^2 = AB^2 + BD^2 - 2AB \,.\, BD \cos ABD,$$

$$\text{or } x^2 = 2^2 + 3^2 - 2\,(2 \times 3) \cos 135^\circ$$

$$= 4 + 9 + 6\sqrt{2} = 21{\cdot}4853;$$

$$\therefore\ x = 4{\cdot}635.$$

3. Let P, Q be the two forces, R their resultant $= Q$ the less.

$$\text{Then } R^2 = P^2 + Q^2 + 2PQ \cos 135^\circ;$$

$$\therefore\ 0 = P^2 - 2PQ \cos 45^\circ;$$

$$\therefore\ \frac{P}{Q} = \frac{2 \cos 45^\circ}{1} = \frac{\sqrt{2}}{1}.$$

5. Let θ be the required $\angle$. Then since

$$R^2 = P^2 + Q^2 + 2PQ\cos\theta;$$

$$\therefore\ 1^2 = 2^2 + 3 + 2\times 2\sqrt{3}\cos\theta = 7 + 4\sqrt{3}\cos\theta;$$

$$\therefore\ \cos\theta = -\frac{6}{4\sqrt{3}} = -\frac{\sqrt{3}}{2} = -\cos 30^\circ = \cos 150^\circ;$$

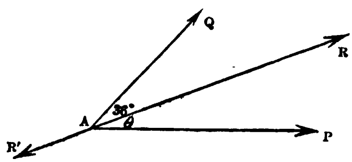
$$\therefore\ \theta = 150^\circ.$$

6. Let $P = 8$, $R = 10$,

$R' = -R$,

required Q, and the

$\angle PAQ = 36^\circ + \theta.$



$$\text{Now}\ R' : P = \sin(36^\circ + \theta) : \sin 36^\circ;$$

$$\therefore\ \sin(36^\circ + \theta) = \frac{10}{8}\sin 36^\circ = \sin 47^\circ\ 17'.$$

$$\text{And}\ Q : R' = \sin\theta : \sin(36^\circ + \theta);$$

$$\therefore\ Q = 10\,\frac{\sin 11^\circ\ 17'}{\sin 47^\circ\ 17'} = 2\cdot663.$$

8. Let $P = (\sqrt{3} + 1)\,a$, $Q = \sqrt{6}a$, $R = 2a$;

$\theta = \angle$ between P and Q, $\phi = \angle$ between P and R.

Then $R^2 = P^2 + Q^2 + 2PQ\cos\theta$;

$$\therefore\ 4 = (4 + 2\sqrt{3}) + 6 + 2(\sqrt{3} + 1)\sqrt{6}\cos\theta;$$

$$\therefore\ \cos\theta = -\frac{2\sqrt{3}(\sqrt{3}+1)}{2\sqrt{6}(\sqrt{3}+1)} = -\frac{1}{\sqrt{2}} = \cos 135^\circ;\ \therefore\ \theta = 135^\circ.$$

Again, $Q^2 = P^2 + R^2 + 2PR\cos\phi$;

$$\therefore\ 6 = (4 + 2\sqrt{3}) + 4 + 2\times 2(\sqrt{3} + 1)\cos\phi;$$

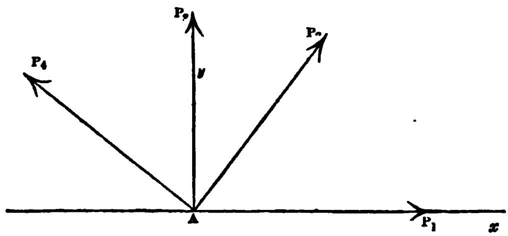
$$\therefore\ \cos\phi = -\frac{2(\sqrt{3}+1)}{4(\sqrt{3}+1)} = -\frac{1}{2} = \cos 120^\circ;\ \therefore\ \phi = 120^\circ.$$

The angle between Q and $R = 360^\circ - (\theta + \phi) = 105^\circ.$

10. Let X be the resultant of all the forces resolved parallel to Ax,

Y . . . . Ay,

R . . . .



$$\begin{aligned} \text{Then } X &= P_1 \cos \alpha + P_2 \cos 60^\circ + P_3 + \cos 90^\circ + P_4 \cos 150^\circ \\ &= 1 \times \cos 0^\circ + 2 \cos 60^\circ + 3 \cos 90^\circ - 4 \cos 30^\circ \\ &= -1{\cdot}466 \end{aligned}$$

$$\begin{array}{rr} 1 \times \cos 0^\circ = & 1{\cdot} \\ 2 \times \cos 60^\circ = & 1{\cdot} \\ 3 \times \cos 90^\circ = & 0{\cdot} \\ \hline & 2{\cdot} \\ -4 \cos 30^\circ = & -3{\cdot}466 \\ \hline \end{array}$$

$$\begin{aligned} Y &= P_1 \sin 0^\circ + P_2 \sin 60^\circ + P_3 \sin 90^\circ + P_4 \sin 150^\circ \\ &= 1 \sin 0^\circ + 2 \sin 60^\circ + 3 \sin 90^\circ + 4 \sin 30^\circ \\ &= 6{\cdot}732; \end{aligned}$$

$$\begin{array}{rl} 1 \sin 0^\circ & = 0 \\ 2 \sin 60^\circ & = 1{\cdot}732 \\ 3 \sin 90^\circ & = 3{\cdot} \\ 4 \sin 30^\circ & = 2{\cdot} \\ \hline & 6{\cdot}732 \end{array}$$

$$\therefore R = (X^2 + Y^2)^{\frac{1}{2}} = \{(-1{\cdot}466)^2 + (6{\cdot}732)^2\}^{\frac{1}{2}} = 6{\cdot}88.$$

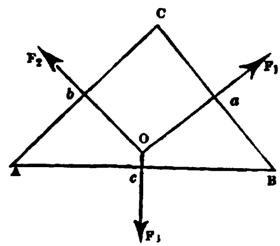
If θ be the inclination of the resultant R to Ax, then

$$\tan \theta = \frac{Y}{X} = -\frac{6{\cdot}732}{1{\cdot}466} = \tan 102^\circ\ 16';$$

$$\therefore \theta = 102^\circ\ 16'.$$

11. Let ABC be the triangle; a, b, c the middle points of its sides. Let F_1, F_2, F_3 be the forces acting at a, b, c in the manner stated in the question. Then because the sides of the

triangle are bisected perpendicularly, the lines F_1a, F_2b, F_3c being produced will meet in O the centre of the circle circumscribing the triangle. We may therefore suppose them to act at O. Then



$$F_1 : F_2 : F_3 :: BC : AC : AB$$
$$:: \sin A : \sin B : \sin C$$
$$:: \sin F_2OF_3 : \sin F_1OF_3 : \sin F_1OF_2;$$

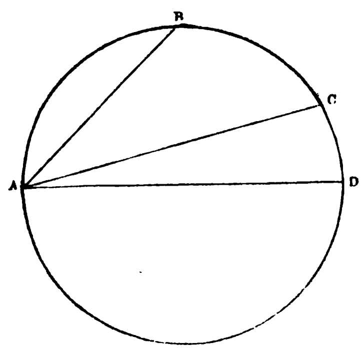
therefore the forces keep each other at rest.

15. $$p^2 + p'^2 + 2pp' \cos \alpha = P^2,$$
$$p^2 + p'^2 - 2pp' \qquad = d^2;$$
$$\therefore\ 2pp'(1 + \cos\alpha) = P^2 - d^2;$$
$$\therefore\ 4pp' = (P^2 - d^2)\sec^2\frac{\alpha}{2};$$
$$\therefore\ p + p' = \left\{P^2\sec^2\frac{\alpha}{2} - d^2\left(\sec^2\frac{\alpha}{2} - 1\right)\right\}^{\frac{1}{2}} = \left(P^2 - d^2\sin^2\frac{\alpha}{2}\right)^{\frac{1}{2}}\sec\frac{\alpha}{2},$$
$$p - p' = d;$$
$$\therefore\ p = \frac{1}{2}\left\{d + \left(P^2 - d^2\sin^2\frac{\alpha}{2}\right)^{\frac{1}{2}}\sec\frac{\alpha}{2}\right\};$$
$$p' = \frac{1}{2}\left\{-d + \left(P^2 - d^2\sin^2\frac{\alpha}{2}\right)^{\frac{1}{2}}\sec\frac{\alpha}{2}\right\}.$$

17. Let

$$\angle DAB = \alpha,$$
$$\angle DAC = \theta;$$
$$AD = d,$$

R the required resultant.



Then

$$AB = d\cos\alpha,$$
$$AC = d\cos\theta;$$

and $R^2 = d^2 \{\cos^2 \alpha + \cos^2 \theta + 2 \cos \alpha \cos \theta \cos (\alpha - \theta)\}$, a maximum.

Differentiating with regard to θ, we obtain

$$-2 \sin \theta \cos \theta - 2 \cos \alpha \{\sin \theta \cos (\alpha - \theta) - \cos \theta \sin (\alpha - \theta)\} = 0;$$

$$\therefore \sin 2\theta = 2 \cos \alpha \sin (\alpha - 2\theta) = \sin 2 (\alpha - \theta) - \sin 2\theta,$$

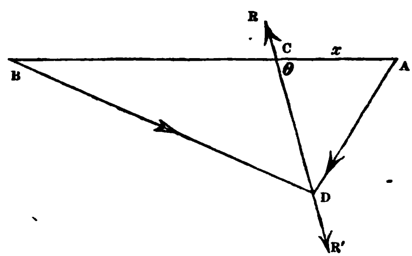
$$2 \sin 2\theta = \sin 2\alpha \cos 2\theta - \cos 2\alpha \sin 2\theta,$$

$$\cot 2\theta = \frac{2 + \cos 2\alpha}{\sin 2\alpha} = \frac{1 + 2 \cos^2 \alpha}{\sin 2\alpha} = \frac{\sin^2 \alpha + 3 \cos^2 \alpha}{2 \sin \alpha \cos \alpha} = \frac{1 + 3 \cot^2 \alpha}{2 \cot \alpha};$$

$$\therefore \cot 2\theta \cot \alpha + 1 = \frac{3}{2} (1 + \cot^2 \alpha), \cot 2\theta - \cot \alpha = \frac{1 + \cot^2 \alpha}{2 \cot \alpha};$$

$$\therefore \cot (\alpha - 2\theta) = 3 \cot \alpha.$$

20. Let AD, BD represent the direction of the forces, and DR the direction of the resultant.



Then $R = \sqrt{P^2 + Q^2} = \sqrt{3^2 + 4^2} = 5.$

Let DR meet AB in C; and let $AC = x$, $\angle ACD = \theta$. Then since the moments about C must be equal, we have

$$P \times AC . \sin DAC = Q \times BC . \sin DBC,$$

$$\text{or } 4x . \frac{\sqrt{3}}{2} = \frac{3}{2} (10 - x),$$

$$\text{whence } x = 3 \cdot 0217.$$

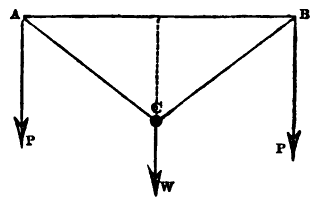
To find θ. If $R' = -R$, we have

$$\frac{R'}{P} = \frac{\sin 90^\circ}{\sin BDC}; \quad \therefore \sin (\theta - 30^\circ) = \frac{4}{5} = \sin 53^\circ\ 7'\ 48'';$$

$$\therefore \theta = 83^\circ\ 7'\ 48''.$$

Ex. 2.

3. The forces as in the figure. When there is equilibrium the tension in the string BC = tension in the string $AC = P$; and these two tensions are balanced by the weight $P \times \sqrt{3}$, which we may regard as their resultant.



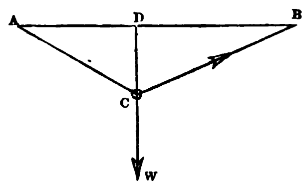
Therefore if θ be the angle ACB, we have

$$3P^2 = P^2 + P^2 + 2P^2 \cos\theta,$$

$$\text{whence } \cos\theta = \frac{1}{2} = \cos 60°;$$

$$\therefore \theta = 60°.$$

6. The forces as in the figure.



The tension in $BC = W$.

Therefore AC produced will bisect the angle BCW.

Let $\angle BAC = \theta$;

$$\text{then } \angle ACB = \frac{\pi}{2} + \theta \text{ and } \angle ABC = \frac{\pi}{2} - 2\theta;$$

$$\therefore AB : AC = \sin\left(\frac{\pi}{2} + \theta\right) : \sin\left(\frac{\pi}{2} - 2\theta\right),$$

$$\text{or } 2 : 1 = \cos\theta : \cos 2\theta;$$

$$\therefore 2\cos 2\theta = \cos\theta,$$

$$\text{whence } 4\cos^2\theta - \cos\theta = 2,$$

which equation solved gives

$$\cos\theta = \frac{1 \pm \sqrt{33}}{8} = \cdot 8430703 = \cos(32° \; 32' \; 3''),$$

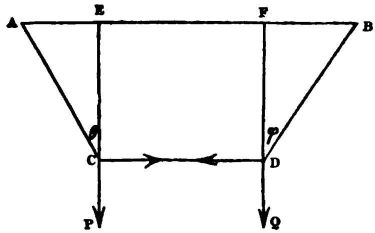
and therefore the remaining angles of the triangle are known, and therefore BC may be found in terms of AB.

9. Let $ACDB = 21$, $AC = 6$, $CD = 7$;

$$\therefore DB = 8, \quad AB = 14.$$

$$\text{If } AE = x, \quad BF = y;$$

T the tension of string CD, $ACE=\theta$, $BDF=\phi$;



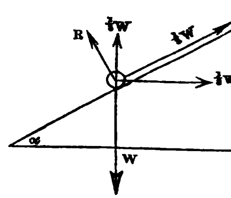
$$\text{then } P : T = \sin(90^\circ+\theta) : \sin(180^\circ-\theta)$$
$$= \cos\theta : \sin\theta = 1 : \tan\theta.$$

$$\text{Similarly } T : Q = \sin\phi : \cos\phi = \tan\phi : 1;$$
$$\therefore P : Q = \tan\phi : \tan\theta = y : x.$$

$$\text{Now } AC^2 - AE^2 = CE^2 = DF^2 = DB^2 - BF^2;$$
$$\therefore 6^2 - x^2 = 8^2 - y^2,$$
$$y^2 - x^2 = 64 - 36 = 28.$$
$$\text{But } y + x = 14 - 7 = 7;$$
$$\therefore y - x \qquad = 4.$$

Hence $2y = 11$, $2x = 3$; and $\therefore P : Q = 11 : 3$.

12. Resolving the forces parallel to the plane, we have



$$\frac{1}{3}W + \frac{1}{3}W\cos\alpha = \frac{2}{3}W\sin\alpha,$$

α being the inclination of the plane;

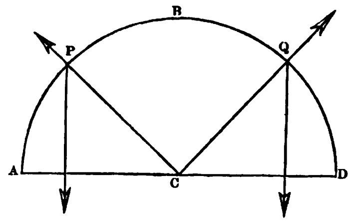
$$\therefore 1 + \cos\alpha = 2\sin\alpha,$$

$$2\cos^2\frac{\alpha}{2} = 4\sin\frac{\alpha}{2}\cos\frac{\alpha}{2},$$

$$\text{whence } \tan\frac{\alpha}{2} = \frac{1}{2} = \tan 26^\circ\ 33'\ 54'';$$

$$\therefore \alpha = 2\tan^{-1}(\cdot 5) = 53^\circ\ 7'\ 48''.$$

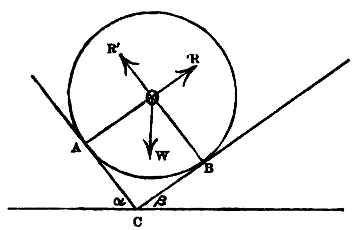
17. Since the string is equal in length to the arc of a quadrant, the angle PCQ will be a right angle. Therefore taking the moments of P and Q about C, we have



$$P \times r \cos PCA = Q \times r \cos QCD = Q \times r \sin PCA,$$

$$\text{whence } \tan PCA = \frac{P}{Q}.$$

19. The reactions R, R' pass through the centre of the sphere whose weight is W; hence the centre O is kept at rest by these three forces;

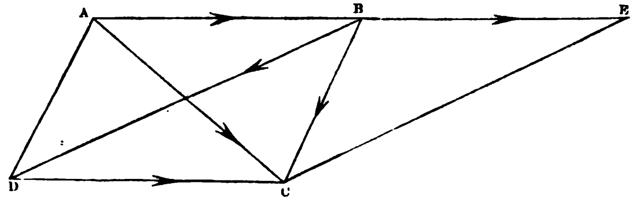


$$\therefore R : W = \sin R'OW : \sin ROR'$$
$$= \sin WOB : \sin AOB$$
$$= \sin \beta : \sin (\alpha + \beta),$$

and $R' : W = \sin ROW : \sin ROR' = \sin \alpha : \sin (\alpha + \beta)$.

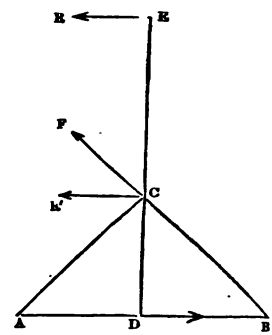
Ex. 3.

1. Produce AB to E, making $BE = AB$, and join CE, CA. The resultant of BE and BD is the force represented by the diagonal BC; therefore the resultant of AB, BD and DC is the



resultant of BC and DC, and is therefore represented by AC. Hence a force represented in magnitude and direction by CA will keep the parallelogram at rest.

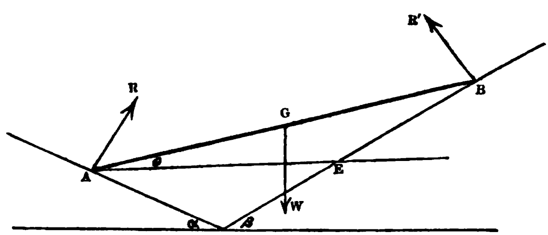
5. Let each of the forces equal P, and R their resultant; and let R_1 be the resultant of the forces in the directions BC, CA; then we have



$$R_1^2 = P^2 + P^2 + 2PP\cos 90° = 2P^2;$$

$\therefore\ R_1 = P\sqrt{2}$ and its direction CR_1 bisects the right angle ACF, and is therefore parallel to AB. Now R is the resultant of R_1 and P in the direction AB; $\therefore R = R_1 - P = P(\sqrt{2}-1)$: and E its point of application is determined by taking moments about E: hence $P\times DE = R_1 \times CE$; $\therefore DE = CE\sqrt{2}$.

8. The beam is kept at rest by three forces R, R', the reactions at A and B; and W the weight of the beam acting at its middle point G.



Resolving these forces horizontally and vertically; then taking the moments about A, we obtain

$$R\sin\alpha = R'\sin\beta,\quad R\cos\alpha + R'\cos\beta = W,$$

$$\text{and } R'\times AB\cos ABE = W\times AG\cos BAE;$$

$$\therefore\ R'\times\cos(\beta-\theta) = \frac{1}{2}W\cos\theta.$$

Eliminating R from the first and second, we get

$$R'\left(\frac{\sin\beta}{\sin\alpha}\cos\alpha + \cos\beta\right) = W;\quad \therefore\ R'\sin(\alpha+\beta) = W\sin\alpha.$$

$$\text{Hence }\frac{\cos(\beta-\theta)}{\cos\theta} = \frac{W}{2R'} = \frac{\sin(\alpha+\beta)}{2\sin\alpha};$$

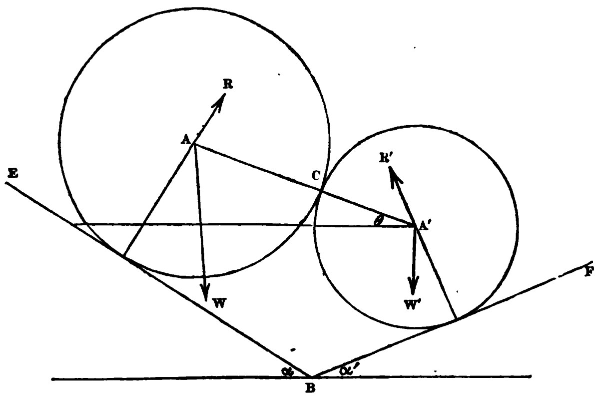
$$\therefore\ \tan\theta = \frac{\sin(\beta-\alpha)}{2\sin\alpha\sin\beta},\quad R = \frac{W\sin\beta}{\sin(\alpha+\beta)};\quad R' = \frac{W\sin\alpha}{\sin(\alpha+\beta)}.$$

9. The sphere A is kept at rest by three forces: viz. its own weight W, the reaction R of the plane, and the reaction M at C of the sphere A'. Resolving these forces parallel to BE, we have

$$M \cos(\alpha - \theta) = W \sin \alpha.$$

Similarly for the forces which keep the sphere A' at rest, we get

$$M \cos(\alpha' + \theta) = W' \sin \alpha'.$$



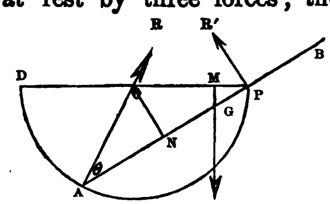
Whence $\dfrac{\cos(\alpha - \theta)}{\cos(\alpha' + \theta)}$ or $\dfrac{\cos \alpha + \sin \alpha \tan \theta}{\cos \alpha' - \sin \alpha' \tan \theta} = \dfrac{W \sin \alpha}{W' \sin \alpha'}$.

$$\therefore \frac{\cot \alpha + \tan \theta}{\cot \alpha' - \tan \theta} = \frac{W}{W'};$$

$$\therefore \tan \theta = \frac{W}{W + W'} (\cot \alpha + \cot \alpha') - \cot \alpha$$

$$= \frac{W \cot \alpha' - W' \cot \alpha}{W + W'}.$$

11. The beam AB is kept at rest by three forces; the reaction R at A, the reaction R' at P, and the weight W at its middle point G.



Let $AB = 3r$, r being the radius of the bowl,

$$AP = x, \ \angle APD = \theta.$$

Resolve parallel to AB and take the moments about P; by this course the solution is not

complicated by the introduction of the unknown reaction R' into the equations.

$$\text{Then } R\cos\theta = W\sin\theta,$$

$$\text{and } Rx\sin\theta = W\left(x-\frac{3r}{2}\right)\cos\theta;$$

$$\therefore\ x\tan\theta = \left(x-\frac{3r}{2}\right)\cot\theta;$$

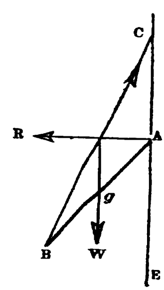
$$\therefore\ \tan^2\theta = 1-\frac{3r}{2x}.$$

$$\text{Now } \cos\theta = \frac{AN}{AO} = \frac{x}{2r};\quad \therefore\ \sec^2\theta = \frac{4r^2}{x^2}.$$

$$\therefore\ 2-\frac{3r}{2x} = \sec^2\theta = \frac{4r^2}{x^2},$$

$$x^2-\frac{3}{4}rx = 2r^2;\ \text{whence } x = 1{\cdot}838r.$$

12. Let $\angle BCA = \theta$, $BAE = \phi$.



Since the beam is uniform, its centre of gravity will be at its middle point. The beam is kept at rest by its own weight, the tension T of the string BC, the reaction R of the wall. Let $CA = x$.

Then, resolving perpendicular to the string, and taking moments about C, we obtain

$$R\cos\theta = W\sin\phi,$$

$$Rx = W\times\frac{1}{2}AB\sin\phi.$$

$$\therefore\ \frac{\cos\theta}{x} = \frac{2}{3}\cdot\frac{\sin\theta}{\sin\phi} = \frac{2}{3}\times\frac{3}{5} = \frac{2}{5}.$$

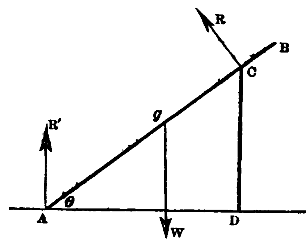
$$\text{But } \cos\theta = \frac{5^2+x^2-3^2}{2\times 5x} = \frac{2x}{5};$$

$$\therefore\ 4x^2 = 16+x^2;\quad \therefore\ x = \frac{4}{\sqrt{3}} = 2{\cdot}3094\text{ ft.}$$

Resolving vertically, $T\cos\theta = W$;

$$\therefore\ T = \frac{5W}{2x} = \frac{5\sqrt{3}}{8}W.$$

14. Let AB represent the beam resting on the vertical prop CD at C, and on the horizontal plane AD. The beam is kept at rest by the reaction (R') of the plane AD; its own weight; the tension of the string AD; and the reaction (R) of the prop CD perpendicular to the beam.



Now since $CD = 3$, and $AD = 4$;

$$\therefore \; AC = 5.$$

Let the $\angle CAD = \theta$. Then, resolving the forces horizontally,

$$T - R\sin\theta = 0; \quad \therefore \; T = R\sin\theta = \frac{3}{5}R.$$

Taking the moments about A, we have

$$R \times AC = W \times Ag\cos\theta,$$

$$\text{or} \quad 5R = 3W\cos\theta = \frac{12W}{5};$$

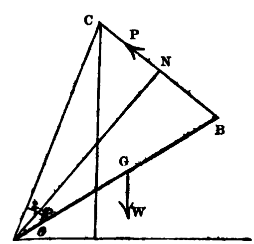
$$\therefore \; R = \frac{12}{25}W;$$

$$\therefore \; T = R \times \frac{3}{5} = \frac{36}{125}W.$$

17. Let θ be the inclination of the beam AB to the horizon,

$$\angle BAC = 2\phi.$$

Resolving the forces which keep the beam at rest perpendicularly to AC,



$$P\cos\phi = W\cos(2\phi + \theta).$$

Taking moments about A,

$$P \times AB\cos\phi = W \times \frac{1}{2}AB\cos\theta.$$

Hence, since $P = \frac{1}{2}W$,

$$\cos\phi = \cos\theta; \quad \therefore \; \phi = \theta;$$

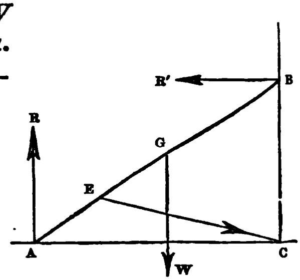
$$\text{and} \; \therefore \; \cos\theta = 2\cos 3\theta = 2(4\cos^3\theta - 3\cos\theta),$$

$$\cos\theta = 0, \text{ and } 8\cos^2\theta - 7 = 0; \quad \therefore \; \cos\theta = \left(\frac{7}{8}\right)^{\frac{1}{2}}.$$

$$\text{Hence } \theta = 90°, \text{ or } \cos^{-1}\left(\frac{7}{8}\right)^{\frac{1}{2}}.$$

N.B. The point G, when the beam is at rest, will be in the vertical through C.

21. The lines and forces as in the figure.



Let $\angle BAC = \alpha$, $\angle ACE = \epsilon$, W equal weight of the beam, and $AB = 2a$.

Then, resolving the forces vertically,

$$R - T\sin\epsilon - W = 0 \text{............} (1).$$

Taking the moments about B,

$$R \times 2a\cos\alpha - T \times 2a\sin\alpha\cos\epsilon - W \times a\cos\alpha = 0 \text{.........} (2).$$

Eliminating R between (1) and (2), we get

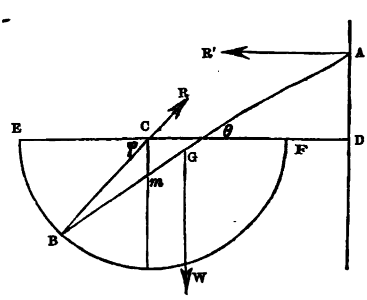
$$T\sin\epsilon + W = \frac{T\sin\alpha . \cos\epsilon}{\cos\alpha} + \frac{1}{2}W;$$

$$\therefore T(\sin\alpha\cos\epsilon - \cos\alpha\sin\epsilon) = \frac{W\cos\alpha}{2},$$

$$T = \frac{W\cos\alpha}{2\sin(\alpha - \epsilon)}.$$

Otherwise. Resolve horizontally and vertically, and take the moments about C.

23. Let AB be the beam resting against the vertical plane at A, and upon the bowl at B.



Let ECF a horizontal diameter of the bowl produced meet the vertical plane at D. Let the radius of the bowl be equal to r; $AB = 2l$; $CD = a$, and W equal the weight of the beam.

Let $\angle BCE = \phi$; and inclination of the beam to the horizon $= \theta$.

The beam is supported by

its weight acting at G; the reaction R' of the vertical plane; and the reaction R in the direction of BC.

Resolving vertically, we have

$$\therefore \quad R\sin\phi - W = 0, \text{ or } R = \frac{W}{\sin\phi}.$$

Taking the moments about A, we have,

$$R\,.\,AB\sin(\phi-\theta) - W\,.\,AG\cos\theta = 0;$$

$$\therefore \frac{\sin(\phi-\theta)}{\sin\phi} - \frac{\cos\theta}{2} = 0;$$

$$\therefore \tan\phi = 2\tan\theta \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

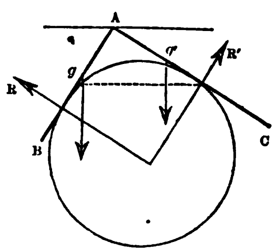
Let Cm be a vertical line meeting AB in m;

$$\frac{\sin BCm}{\sin BmC} = \frac{\cos\phi}{\cos\theta} = \frac{Bm}{BC} = \frac{AB - Am}{r}$$

$$= \frac{2l - a\,.\sec\theta}{r};$$

$$\therefore 2l\cos\theta = r\cos\phi + a \ldots\ldots\ldots\ldots\ldots(2).$$

27. The forces as in the figure.



Since the arms of the lever are uniform, their weights are proportional to their lengths.

The lever is kept at rest by the weights of its own arms and the reactions of the circle at the points of contact.

Let θ be the inclination of the arm $AC\,(=2a)$ to the horizon.

Resolving the forces in direction of the arm AC, we have

$$R - (a+b)\sin\theta = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

Taking the moments about the point of contact of *the same* arm, we get

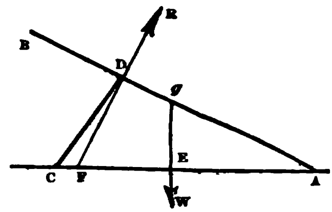
$$R\times c - b(c\cos\theta + b\sin\theta) - a(c-a)\cos\theta = 0 \ldots\ldots\ldots\ (2).$$

Eliminating R by means of (1) and (2),

$$c(a+b)\sin\theta = bc\cos\theta + b^2\sin\theta + ac\cos\theta - a^2\cos\theta,$$

$$\text{whence } \tan\theta = \frac{bc + ac - a^2}{bc + ac - b^2}.$$

31. Let AB be the beam leaning upon the prop CD. Let $AB = 2a$, $\angle CAD = \alpha$, $\angle ACD = \beta$. The beam is supported by its weight (W) acting at its middle point g; and the reaction R of the prop at right angles to the beam.



Taking the moments about A, we have

$$R \, . \, AD - W \, . \, AE = 0.$$

$$\text{Now } AD : CD = \sin\beta : \sin\alpha; \quad \therefore \; AD = \frac{b \sin\beta}{\sin\alpha};$$

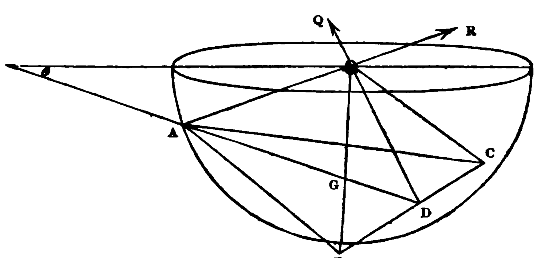
$$\therefore \; R \frac{b \sin\beta}{\sin\alpha} = W \, . \, a \cos\alpha;$$

$$\therefore \; R = \frac{W \, . \, a \sin\alpha \cos\alpha}{b \sin\beta} = \frac{W \, . \, a \sin 2\alpha}{2b \sin\beta},$$

and the resolved part of this pressure in a direction perpendicular to DC, or the strain upon the prop is

$$\frac{W a \sin 2\alpha}{2b \sin\beta} \sin CDF = \frac{W a \sin 2\alpha \cos(\alpha + \beta)}{2b \sin\beta}.$$

32. Let ABC be the isosceles triangle resting with its angular points in contact with the bowl; $AB = AC = a$, $AD = h$; radius of bowl $= r$; G the centre of gravity of the



triangle, and therefore $DG = \frac{1}{3} h$. Let θ measure the inclination of the triangle to the horizon; $\angle DAO = \alpha$, $ADO = \beta$.

The triangle is kept at rest, (1) by reactions at its angular points, which act in the directions of normals to the bowl at A, B, C, and therefore pass through the centre O, (2) by its own weight acting vertically at G. The reactions at B and C being equal to one another, may be replaced by their resultant (Q) which bisects BC in D. Let R be the reaction at A.

Resolving the forces horizontally, and taking moments about G, we get

$$R\cos(\alpha-\theta)=Q\cos(\beta+\theta),$$

$$R\times GA\sin\alpha=Q\times GD\sin\beta;\quad\therefore\ 2R\sin\alpha=Q\sin\beta.$$

$$\text{Hence}\quad\frac{\cos\alpha\cot\theta+\sin\alpha}{2\sin\alpha}=\frac{\cos\beta\cot\theta-\sin\beta}{\sin\beta};$$

$$\therefore\ \cot\theta=\frac{3}{2}\div\left(\frac{\cos\beta}{\sin\beta}-\frac{\cos\alpha}{2\sin\alpha}\right)=\frac{3}{2\cot\beta-\cot\alpha}.$$

Now $OD^2=OB^2-BD^2=r^2-(a^2-h^2)$,

$$\cos\beta=\frac{AD^2+DO^2-AO^2}{2AD+DO}=\frac{h^2+(r^2-a^2+h^2)-r^2}{2h\,(r^2-a^2+h^2)^{\frac{1}{2}}}=\frac{2h^2-a^2}{2h\,(r^2-a^2+h^2)^{\frac{1}{2}}},$$

$$\sin\beta=\frac{\{4h^2(r^2-a^2+h^2)-(4h^4-4a^2h^2+a^4)\}^{\frac{1}{2}}}{2h\,(r^2-a^2+h^2)^{\frac{1}{2}}}=\frac{(4r^2h^2-a^4)^{\frac{1}{2}}}{2h\,(r^2-a^2+h^2)^{\frac{1}{2}}}.$$

$$\text{Also}\quad\cos\alpha=\frac{r^2+h^2-(r^2-a^2+h^2)}{2rh}=\frac{a^2}{2rh},$$

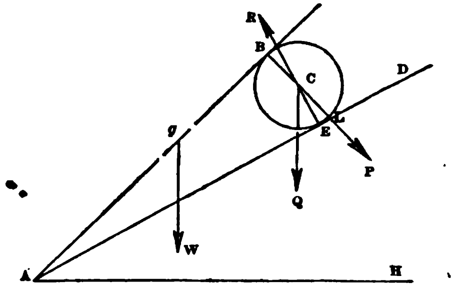
$$\sin\alpha=\frac{(4r^2h^2-a^4)^{\frac{1}{2}}}{2rh};$$

$$\therefore\ 2\cot\beta-\cot\alpha=\frac{(2h^2-a^2)}{(4r^2h^2-a^4)^{\frac{1}{2}}}-\frac{a^2}{(4r^2h^2-a^4)^{\frac{1}{2}}}=\frac{4h^2-3a^2}{(4r^2h^2-a^4)^{\frac{1}{2}}};$$

$$\therefore\ \cot\theta=\frac{3\,(4r^2h^2-a^4)^{\frac{1}{2}}}{4h^2-3a^2}.$$

34. Let AB be the beam moveable about A; W its weight acting at its centre of gravity g.

B the point of contact of the beam and sphere, of which the centre is C, and weight Q (= 4 lbs.)



The sphere is in equilibrium from the reaction (R) of the plane at the point of contact; from the pressure (P) of the beam at B upon the sphere; and from its own weight. These three forces all act through the centre (C) of the sphere.

The data are

$$Ag = 42 \text{ in.} = a \text{ suppose}; \quad BC = 9 \text{ in.}, \angle BAD = 15^\circ,$$

$$\angle DAE = 30^\circ;$$

$$\therefore \; CL = \frac{9}{\cos 15^\circ} = 9{\cdot}3177; \quad BL = 18{\cdot}3177,$$

and from the right-angled triangle ALB we have

$$\frac{BL}{AB} = \tan 15^\circ; \quad \therefore \; AB = \frac{LB}{\tan 15^\circ} = 68{\cdot}3751 = b \text{ suppose.}$$

For the condition of equilibrium of the beam, taking the moments about A, we have

$$P \,.\, AB = W \,.\, Ag \cos 45^\circ;$$

$$\therefore \; P = W \,.\, \frac{a}{b} \cos 45^\circ \text{ .... } (1).$$

For the condition of equilibrium of the sphere, resolving the forces in the direction AD, we have

$$Q \sin 30^\circ - P \sin 15^\circ = 0,$$

$$\text{or } 2 - P \sin 15^\circ = 0.$$

Therefore from (1) we have

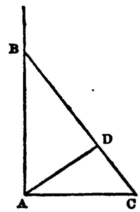
$$2 - W \,.\, \frac{a}{b} \sin 15^\circ \cos 45^\circ = 0,$$

or, substituting for a and b their values

$$W = 2 \times 68 \cdot 3751 \times \frac{1}{42 \times \cdot 2588 \times \cdot 7071}$$

$$= \frac{136 \cdot 7502}{7 \cdot 6859} = 17 \cdot 79 \text{ lbs. nearly.}$$

37. Let B be the point at which the rope must be tied. Let $\angle BCA = \theta$. From A draw AD perpendicular to BC, the direction of the man's pull (P).



The moment of P about $A = P \times AD$

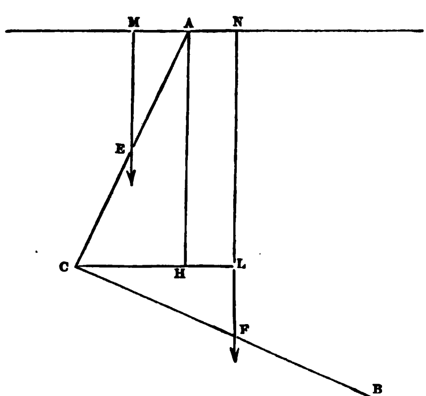
$$= PAB \cos\theta = P \,.\, a \sin\theta \cos\theta = \frac{1}{2} P \,.\, a \sin 2\theta,$$

and this is greatest when $2\theta = 90°$ or $\theta = 45°$.

Hence $AB = BC \sin 45° = \dfrac{a}{\sqrt{2}}$.

38. Let $AC = 2a$, $CB = 2b$, $\angle ACB = 90°$,

$\angle CAM = \theta$ the inclination of CA to the horizon.



The arms being uniform, their weights are proportional to their lengths, and may be supposed to act at their middle points E, F respectively.

Taking moments about A we have

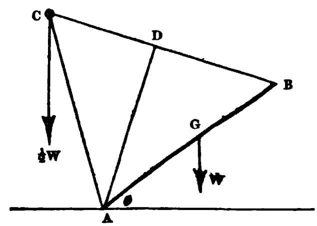
$$2a \times AM = 2b \times AN.$$

Now $AM = a \cos\theta$, and $AN = CL - CH = b \sin\theta - 2a \cos\theta$;

$$\therefore\ a^2 \cos\theta = b^2 \sin\theta - 2ab \cos\theta;$$

$$\therefore\ \tan\theta = \frac{a^2 + 2ab}{b^2}, \text{ or } \theta = \tan^{-1}\frac{a^2+2ab}{b^2}.$$

41. Let AB be the beam; G its centre of gravity at its middle point, and θ its inclination to the horizon.



The beam is kept at rest by the forces,

(1) the reaction at the hinge A,

(2) the weight (W) of the beam at G,

(3) the tension ($= \frac{1}{2}W$) of the string BC.

Take the moments about A, and let $BAC = 2\phi$.

$$\text{Then } W \times \frac{1}{2} AB \cos\theta = \frac{1}{2} W \times AB \cos\phi;$$

$$\therefore\ \cos\theta = \cos\phi; \quad \therefore\ \theta = \phi.$$

Hence when AC is vertical, $\theta + 2\phi = 90°$, and $\therefore\ \theta = 30°$.

44. Taking moments about C we have

$$14\, CB \cos\theta + 2\, CA \cos\theta = 6\, CN.$$

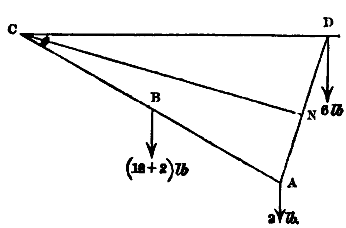


Fig. 1.

Hence in the 1st case (fig. 1),

$$9 \cos \theta = 6 \cos \frac{\theta}{2}; \text{ whence } \theta = 53^\circ\ 27'\ 23''.$$

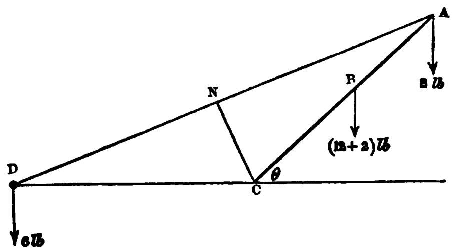


Fig. 2.

And in the 2d case (fig. 2),

$$9 \cos \theta = 6 \sin \frac{\theta}{2}; \text{ whence } \theta = 68^\circ\ 5'\ 10''.$$

FORCES WHICH DO NOT ACT IN THE SAME PLANE.

Ex. 4.

1. Let R, R', R'' denote the reactions at the props; then

$$R + R' + R'' = W.$$

Taking the sides containing the right angle for axes of x and y, and resolving parallel to the axis of z, we have

$$3R' = Wx, \text{ and } 4R'' = Wy.$$

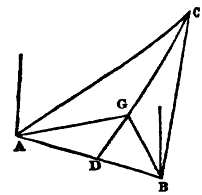
$$\begin{aligned} \text{Again,} \quad R' : R &= 4 : 5, \\ R : R'' &= 5 : 3; \\ \therefore\ R' : R'' &= 4 : 3; \\ \therefore\ Wx = Wy &\text{ and } x = y. \end{aligned}$$

$$\text{Also } R\left(1 + \frac{4}{5} + \frac{3}{5}\right) = W; \quad \therefore R = \frac{5}{12} W;$$

$$\therefore R' = \frac{4}{5} R; \quad \therefore \frac{Wx}{3} = \frac{4}{5}\left(\frac{5}{12} W\right);$$

$$\therefore x = 1 = y.$$

3. The weight of the triangle acts at its centre of gravity G. Join CG, and produce it to meet AB in D.



Let $CG = a$; $DB = p = DB$.

The weight of the triangle acts vertically; resolve it into two vertical forces at C, D;

$$\text{the resolved force at } C = W.\frac{DG}{CD} = \frac{1}{3}W \ldots\ldots\ldots\ldots\ldots (1).$$

Similarly the resolved force at $D = W.\dfrac{CG}{CD} = \dfrac{2}{3}W$; let this be again resolved into two forces acting in the directions of the strings at A and B; then

$$\text{tension of string at } B = \frac{2}{3}W \times \frac{1}{2} = \frac{1}{3}W \ldots\ldots\ldots\ldots (2),$$

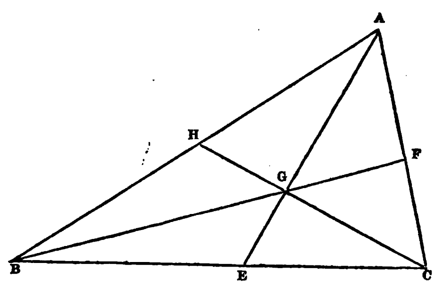
$$\text{and } \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots A = \frac{2}{3}W \times \frac{1}{2} = \frac{1}{3}W \ldots\ldots\ldots\ldots (3).$$

Equations (1), (2), (3) shew the truth of the enunciation.

The Centre of Gravity.

Ex. 5.

1. Bisect BC, CA, AB in E, F, H respectively. Then will AE, BF, CH intersect in G, the centre of gravity.



Now $GB^2 + GC^2 = 2GE^2 + 2BE^2$

$$= \frac{1}{2}GA^2 + \frac{1}{2}BC^2, \text{ since } GE = \frac{1}{2}GA \text{ and } BE = \frac{1}{2}BC.$$

$$\text{Similarly } GC^2 + GA^2 = \frac{1}{2} GB^2 + \frac{1}{2} AC^2,$$

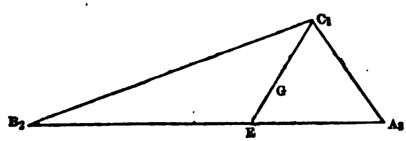
$$\text{and } GA^2 + GB^2 = \frac{1}{2} GC^2 + \frac{1}{2} AB^2;$$

$$\therefore \left(2 - \frac{1}{2}\right)(GA^2 + GB^2 + GC^2) = \frac{1}{2}(AB^2 + AC^2 + BC^2),$$

$$\text{or } 3(GA^2 + GB^2 + GC^2) = AB^2 + AC^2 + BC^2.$$

5. Let $AB = 5$, $BC = 4$, $CA = 2$;

the weights at A, B, C are proportional to 3, 2, 1, respectively.



Let E be the centre of gravity of A and B.

$$\text{Then } BE : EA = 3 : 2.$$

$$\textit{Componendo}, \ BE : BA = 3 : 5;$$

$$\therefore BE = 3,$$

$$CE^2 = BC^2 + BE^2 - 2BC . BC \cos B$$

$$= 16 + 9 - 2 \times 4 \times 3 \left(\frac{4^2 + 5^2 - 2^2}{2 \times 4 \times 5}\right)$$

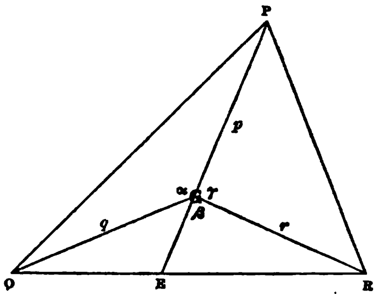
$$= 25 - \frac{3}{5} \times 37 = 25 - 22{\cdot}2 = 2{\cdot}8;$$

$$\therefore CE = 1{\cdot}67332.$$

If G be the centre of gravity of 5 at E and 1 at C; then G is the centre of gravity of the 3 weights at A, B, C; and

$$CG : EG = 5 : 1; \quad \therefore CG = CE \times \frac{5}{6} = \frac{16{\cdot}7332}{12} = 1{\cdot}3944.$$

6. Let G be the centre of gravity of the weights P, Q, R, whose distances from G are p, q, r respectively; produce PG to meet QR in E, then E is the centre of gravity of Q and R; hence



$$P : Q+R = EG : GP;$$

$$\therefore P : P+Q+R = EG : EP$$
$$= \Delta\, QGR : \Delta\, QPR.$$

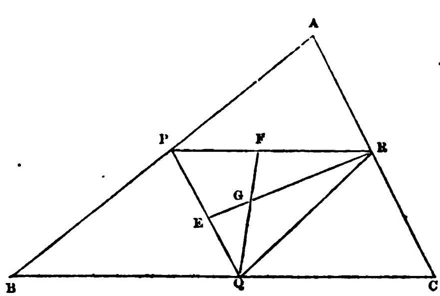
Similarly

$$Q : P+Q+R = \Delta\, RGP : \Delta\, RQP,$$

$$\text{and } R : P+Q+R = \Delta\, PGQ : \Delta\, PRQ.$$

$$\text{Hence } P : Q : R = \Delta\, QGR : \Delta\, RGP : \Delta\, PGQ$$
$$= qr \sin\beta : pr \sin\gamma : pq \sin\alpha$$
$$= \frac{\sin\beta}{p} : \frac{\sin\gamma}{q} : \frac{\sin\alpha}{r}.$$

7. Let P, Q, R express the weights of the sides AB, BC, CA respectively. Join PQ, QR, RP; in PQ take E the centre of gravity of P and Q, and in PR take F the centre of gravity of P and R.



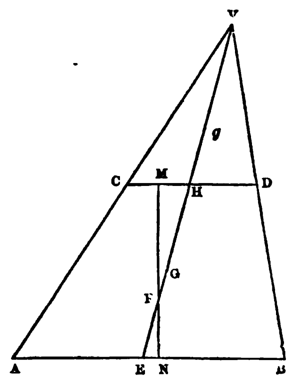
Then $PE : EQ = Q : P = BC : AB = PR : RQ$; hence the line ER bisects the angle PRQ.

Similarly, the line FQ bisects the angle PQR; therefore their point of intersection is the centre of the circle inscribed in the ΔPQR.

Now the centre of gravity of P, Q, R is in the line ER, and also in the line FQ, and is therefore at G the point of their intersection. Hence the centre of the circle inscribed in the ΔPQR is the centre of gravity of the perimeter of ABC.

12. Let $AB = a$, $CD = b$; produce AC, BD to meet.

Bisect AB in E, and join VE cutting CD in H.



Let G, F, g be the centres of gravity of the triangle VAB, trapezoid AD, and triangle VCD respectively;

$$EH = h.$$

Then $GF : Gg = \Delta VCD : \text{trap}^d. AD$

$$= b^2 : a^2 - b^2,$$

and $Gg = VG - Vg$

$$= \frac{2}{3}(VE - VH) = \frac{2}{3}h.$$

Also $VE : VH = a : b$;

$$\therefore VE : EH = a : a - b;$$

$$\therefore EG = \frac{1}{3}EV = \frac{h}{3} \cdot \frac{a}{a-b}.$$

Hence $EF = EG - GF = \dfrac{h}{3}\left(\dfrac{a}{a-b} - \dfrac{2b^2}{a^2-b^2}\right) = \dfrac{h}{3}\left(\dfrac{a+2b}{a+b}\right)$,

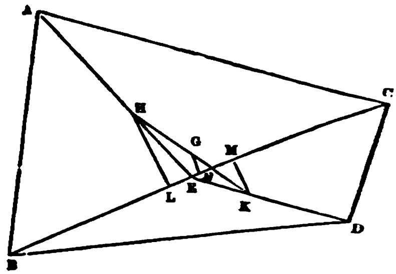
and $FH = h\left\{1 - \dfrac{a+2b}{3(a+b)}\right\} = \dfrac{h}{3}\left(\dfrac{2a+b}{a+b}\right)$.

Through F draw NFM perpendicular to AB and CD;

then $NF : FM = EF : FH = a + 2b : 2a + b$.

17. Bisect the diagonal $BC = p$ in E; join AE, ED; take $EH = \frac{1}{3}EA$, $EK = \frac{1}{3}ED$; draw HL, KM perpendiculars on

BC. Join *HK*, and let *G* be the centre of gravity of the quadrilateral *ABCD*; draw *GN* perpendicular to *BC*.

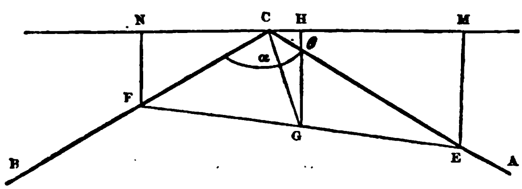


Then quad. $ABCD \times NG = \Delta\, ABC \times HL - \Delta\, BCD \times MK$;

$$\therefore\ \frac{1}{2}p(d+d')\,NG = \frac{1}{2}pd \times \frac{d}{3} - \frac{1}{2}pd' \times \frac{d'}{3};$$

$$\therefore\ NG = \frac{1}{3}\left(\frac{d^2-d'^2}{d+d'}\right) = \frac{1}{3}(d-d').$$

Let the axis of x be horizontal, and that of y vertical; let $CH=\bar{x}$, $HG=\bar{y}$ be the co-ordinates of G, the centre of gravity $CA=a$, $CB=b$.



Then if $MCA=\theta$, $ACB=\alpha$, we have

$$\bar{x}(a+b) = a\,.\,\frac{a}{2}\cos\theta - b\,.\,\frac{b}{2}\cos\{180^\circ - (\alpha+\theta)\}$$

$$= \frac{a^2}{2}\cos\theta + \frac{b^2}{2}\cos(\alpha+\theta),$$

$$\bar{y}(a+b) = a\,.\,\frac{a}{2}\sin\theta + b\,.\,\frac{b}{2}\sin\{180^\circ - (\alpha+\theta)\}$$

$$= \frac{a^2}{2}\sin\theta + \frac{b^2}{2}\sin(\alpha+\theta).$$

Hence $CG = \sqrt{\bar{x}^2 + \bar{y}^2}$

$$= \frac{[\{a^2 \cos\theta + b^2 \cos(\alpha+\theta)\}^2 + \{a^2 \sin\theta + b^2 \sin(\alpha+\theta)\}^2]^{\frac{1}{2}}}{2(a+b)}$$

$$= \frac{[a^4 + b^4 + 2a^2b^2\{\cos\theta\cos(\alpha+\theta) + \sin\theta\sin(\alpha+\theta)\}]^{\frac{1}{2}}}{2(a+b)}$$

$$= \frac{(a^4 - b^4 + 2a^2b^2\cos\alpha)^{\frac{1}{2}}}{2(a+b)}.$$

22. Let L, M, N be the forces represented by magnitude and direction by PA, PB, PC respectively,

$$PA = h,\quad PB = k,\quad PC = l;$$

$$\angle\ CPA = \alpha,\quad CPB = \beta;$$

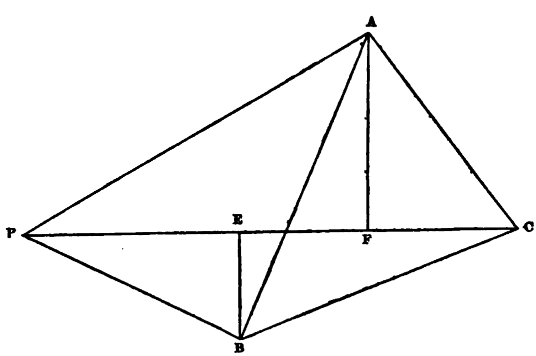
$\bar{x}$, $\bar{y}$ the co-ordinates of the centre of gravity of the $\Delta\, ABC$; P being the origin, and PC the axis of k.

The centre of gravity of ABC coincides with the centre of gravity of 3 equal weights (m) placed at the points A, B, C; hence

$$3m\bar{x} = mh\cos\alpha + mk\cos\beta + ml,$$

$$3m\bar{y} = mh\sin\alpha - mk\sin\beta.$$

To find the direction of the resultant R of the forces L, M, N; we have



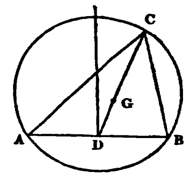
$$R\sin\theta = L\sin\alpha - M\sin\beta,$$

$$R\cos\theta = L\cos\alpha + M\cos\beta + N;$$

$$\therefore \tan\theta = \frac{L\sin\alpha - M\sin\beta}{L\cos\alpha + M\cos\beta + N} = \frac{h\sin\alpha - k\sin\beta}{h\cos\alpha + k\cos\beta + l} = \frac{\bar{y}}{\bar{x}}.$$

Hence the resultant passes through the centre of gravity of ABC.

23. Let AB be the given chord in the circle ABC; ACB any triangle inscribed in the circle. Bisect AB and D; and take D as origin of rectangular co-ordinates.



Let x', y' be the co-ordinates of the point C; (x, y) those of the centre of gravity G; $(\beta, 0)$ those of the centre of the circle.

For the circle we have

$$x'^2 + (y' - \beta)^2 = r^2 \dots\dots\dots\dots\dots\dots (1),$$

and, by similar triangles,

$$x = \frac{1}{3}x', \quad y = \frac{1}{3}y';$$

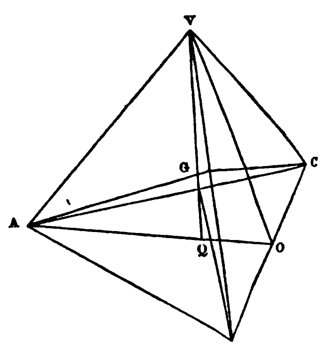
therefore substituting in (1) we get

$$9x^2 + (3y - \beta)^2 = r^2,$$

$$\text{or } x^2 + \left(y - \frac{\beta}{3}\right)^2 = \left(\frac{1}{3}r\right)^2,$$

for the required locus, which is a circle whose radius is $\frac{1}{3}r$.

26. Let V be the vertex, and ABC the base of any triangular pyramid. Bisect BC in O; join AO, and divide it in Q so that $OQ = \frac{1}{3}OA$; join QV, and take $QG = \frac{1}{4}QV$; then G is the centre of gravity of the pyramid.



Join GA, GB, and GC; also VO.

Now

$$2VO^2 + 2BO^2 = VB^2 + VC^2,$$

$$\text{and } BO = \frac{1}{2}BC;$$

$$\therefore\ 2VO^2 = VB^2 + VC^2 - \frac{1}{2}BC^2;$$

$$\text{also } 2AO^2 = AB^2 + AC^2 - \frac{1}{2}BC^2;$$

$$\text{and } VA^2 = AQ^2 + VQ^2 - 2AQ\,.\,VQ\cos AQV,$$
$$VO^2 = OQ^2 + VQ^2 + 2OQ\,.\,VQ\cos AQV;$$

$$\therefore\ VA^2 + 2VO^2 = AQ^2 + 2OQ^2 + 3VQ^2;\ \text{since } AQ = 2QO.$$

Hence $VA^2 + VB^2 + VC^2 - \frac{1}{2}BC^2$

$$= 6OQ^2 + 3\left(\frac{4}{3}VG\right)^2 = \frac{2}{3}AO^2 + \frac{16}{3}GV^2$$

$$= \frac{1}{3}\left(AB^2 + AC^2 - \frac{1}{2}BC^2\right) + \frac{16}{3}GV^2;$$

$$\therefore\ 3(VA^2 + VB^2 + VC^2) = AB^2 + AC^2 + BC^2 + 16GV^2.$$

Similarly,

$$3(AV^2 + AB^2 + AC^2) = VB^2 + VC^2 + BC^2 + 16GA^2,$$
$$3(BA^2 + BV^2 + BC^2) = AV^2 + AC^2 + VC^2 + 16GB^2,$$
$$\text{and } 3(CA^2 + CB^2 + CV^2) = AB^2 + AV^2 + BV^2 + 16GC^2,$$

add and divide by 4, then will

$$4(GA^2 + GB^2 + GC^2 + GV^2)$$
$$= AB^2 + AC^2 + BC^2 + AV^2 + BV^2 + CV^2.$$

32. The base of the pyramid cut off will be an equilateral triangle; and if a be each of the equal edges of the cube, the diagonal of one face,—or a side of the base of the pyramid,—will be equal to $a\sqrt{2}$, and the diagonal of the cube $= a\sqrt{3}$; also the altitude of the pyramid $= \frac{a}{\sqrt{3}}$.

$\frac{a\sqrt{3}}{2}$ = dist. of cent. grav. of cube from the angle opposite to that cut off.

$\frac{3a\sqrt{3}}{4}$ = dist. of cent. grav. of pyramid ....

Let x = dist. of cent. grav. of remaining portion from the same angle, the volume of the pyramid $=\dfrac{a^3}{6}$.

$$\therefore \frac{5a^3}{6}x=\frac{a^4\sqrt{3}}{2}-\frac{a^4\sqrt{3}}{8},$$

$$\text{or } \frac{5x}{6}=\frac{4a\sqrt{3}-a\sqrt{3}}{8}=\frac{3a\sqrt{3}}{8};$$

$$\therefore x=\frac{18a\sqrt{3}}{40}=\frac{9\sqrt{3}}{20}a.$$

33. Let a denote an edge of the cube,

$$\text{then } a\sqrt{3}=\text{a diagonal},$$

$$\text{and } a : a\sqrt{2}=\frac{a}{2} : \text{side of base of pyramid}=\frac{a}{\sqrt{2}},$$

$$\text{and } \left(\frac{a^2}{4}-\frac{a^2}{6}\right)^{\frac{1}{2}}=\left(\frac{2a^2}{24}\right)^{\frac{1}{2}}=\frac{a}{2\sqrt{3}}=\text{alt. of pyramid};$$

$$\therefore \frac{3}{4}\cdot\frac{a}{2\sqrt{3}}=\frac{3a}{8\sqrt{3}}=\text{dist. of cent. grav. of pyramid from the angle cut off};$$

$$\therefore a\sqrt{3}-\frac{3a}{8\sqrt{3}}=\frac{21a}{8\sqrt{3}}=\text{dist. of cent. of grav. of pyramid from angle of cube opposite to the one cut off.}$$

Let x = dist. of cent. grav. of remaining portion from the same angle,

$$\text{volume of pyramid}=\frac{1}{3}\cdot\frac{a^2\sqrt{3}}{8}\cdot\frac{a}{2\sqrt{3}}=\frac{a^3}{48};$$

$$\therefore \text{volume of remaining portion}=\frac{47a}{48};$$

$$\therefore \frac{47a^3}{48}x=a^3\frac{a\sqrt{3}}{2}-\frac{a^3}{48}\cdot\frac{21a}{8\sqrt{3}};$$

$$\therefore \frac{47}{48}x=\frac{a\sqrt{3}}{2}-\frac{7a}{16\times 8\sqrt{3}}=\frac{185a}{16\times 8\sqrt{3}},$$

$$\text{whence } x=\frac{185\sqrt{3}}{376}a.$$

Ex. 6.

1. The area being symmetrical, the formula to be used is

$$\bar{x} = \frac{\int xydx}{\int ydx},$$

and from the equation to the curve, we have

$$y = (a^2 - x^2)^{\frac{1}{2}};$$

$$\therefore \frac{\int xydx}{\int ydx} = \frac{\int (a^2 - x^2)^{\frac{1}{2}} xdx}{\int (a^2 - a^2)^{\frac{1}{2}} dx}.$$

Now $\int_0^a (a^2 - x^2)^{\frac{1}{2}} dx = \frac{x}{2}(a^2 - x^2)^{\frac{1}{2}} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} = \frac{\pi a^2}{4}$

between limits $x = 0$, $x = a$;

$$\int_0^a (a^2 - x^2)^{\frac{1}{2}} xdx = -\frac{1}{3}(a^2 - x^2)^{\frac{3}{2}} = \frac{1}{3}a^3, \text{ from } x = 0 \text{ to } x = a;$$

$$\therefore \bar{x} = \frac{1}{3}a^3 \div \frac{a^2\pi}{4} = \frac{4a}{3\pi}.$$

5. Since $y = a \operatorname{vers}^{-1}\frac{x}{a} + (2ax - x^2)^{\frac{1}{2}}$;

$$\therefore \int xydx = a\int x \operatorname{vers}^{-1}\frac{x}{a}dx + \int x(2ax - x^2)^{\frac{1}{2}}dx.$$

Now $\int_0^{2a} x \operatorname{vers}^{-1}\frac{x}{a}dx = \frac{5\pi a^2}{4}$, and $\int_0^{2a} x(2ax - x^2)^{\frac{1}{2}}dx = \frac{\pi a^3}{2}$;

$$\therefore \int_0^{2a} xydx = \frac{5}{4}\pi a^3 + \frac{1}{2}\pi a^3 = \frac{7}{4}\pi a^3.$$

Similarly $\int_0^{2a} ydx = a\int_0^{2a} \operatorname{vers}^{-1}\frac{x}{a}dx + \int_0^{2a}(2ax - x^2)^{\frac{1}{2}}dx$

$$= a \times \pi a + \frac{1}{2}\pi a^2 = \frac{3}{2}\pi a^2.$$

Hence $\bar{x} = \dfrac{\int_0^{2a} xydx}{\int_0^{2a} ydx} = \frac{7}{4}\pi a^3 \div \frac{3}{2}\pi a^2 = \frac{7}{6}a.$

7. $$x = \frac{c}{2}\left\{\epsilon^{\frac{y}{c}} + \epsilon^{-\frac{y}{c}}\right\}; \quad \therefore \epsilon^{\frac{y}{c}} + \frac{1}{\epsilon^{\frac{y}{c}}} = \frac{2x}{c};$$

$$\therefore \epsilon^{\frac{2y}{c}} - \frac{2x}{c}\epsilon^{\frac{y}{c}} + \frac{x^2}{c^2} = \frac{x^2 - c^2}{c^2};$$

$$\therefore \epsilon^{\frac{y}{c}} - \frac{x}{c} = \frac{(x^2-c^2)^{\frac{1}{2}}}{c}; \quad \therefore \epsilon^{\frac{y}{c}} = \frac{x + (x^2-c^2)^{\frac{1}{2}}}{c},$$

whence $y = c\log\left\{\frac{x+(x^2-c^2)^{\frac{1}{2}}}{c}\right\}$.

Therefore $$\bar{x} = \frac{\int xydx}{\int ydx} = \frac{c\int_c^{\frac{5}{4}c}\log\left\{\frac{x+(x^2-c^2)^{\frac{1}{2}}}{c}\right\}xdx}{c\int_c^{\frac{5}{4}c}\log\left\{\frac{x+(x^2-c^2)^{\frac{1}{2}}}{c}\right\}dx}.$$

$$\int\log\left\{\frac{x+(x^2-c^2)^{\frac{1}{2}}}{c}\right\}xdx = \frac{1}{2}x^2\log\left\{\frac{x+(x^2-c^2)^{\frac{1}{2}}}{c}\right\} - \frac{1}{2}\int\frac{x^2dx}{(x^2-c^2)^{\frac{1}{2}}}$$

$$= \frac{1}{2}x^2\log\left\{\frac{x+(x^2-c^2)^{\frac{1}{2}}}{c}\right\} - \frac{x}{4}(x^2-c^2)^{\frac{1}{2}} - \frac{c^2}{4}\log\{x+(x^2-c^2)^{\frac{1}{2}}\};$$

$$\therefore \int_c^{\frac{5}{4}c}\log\left\{\frac{x+(x^2-c^2)^{\frac{1}{2}}}{c}\right\}xdx = \frac{25c^2}{32}\log_e 2 - \frac{15c^2}{64} - \frac{c^2}{4}\log_e 2$$

$$= \frac{c^2}{64}\{34\log_e 2 - 15\}.$$

Again,

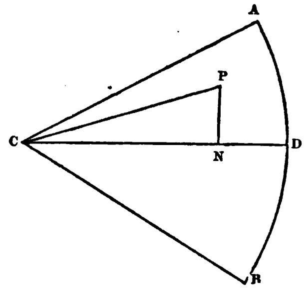
$$\int\log\left\{\frac{x+(x^2-c^2)^{\frac{1}{2}}}{c}\right\}dx = x\log\left\{\frac{x+(x^2-c^2)^{\frac{1}{2}}}{c}\right\} - (x^2-c^2)^{\frac{1}{2}}$$

between limits $= \frac{5c}{4}\log_e 2 - \frac{3c}{4} = \frac{c}{4}(5\log_e 2 - 3)$;

$$\therefore \bar{x} = \frac{c^2}{64}(34\log_e 2 - 15) \div \frac{c}{4}(5\log_e 2 - 3)$$

$$= \frac{c}{16}\left(\frac{34\log_e 2 - 15}{5\log_e 2 - 3}\right).$$

8. Let C be the centre of the circle of which ABC is a sector. From C draw the straight line CD bisecting the arc AB.



Let $CD = c$, $\angle DCA = \alpha$.

Then if P be any point in the area of the sector,

CAB and $CP = r$,

$\angle DCP = \theta$,

we shall have

$$\bar{x} = \frac{\int_{-\alpha}^{\alpha}\int_0^{c} r^2 \cos\theta \,.\, d\theta \,.\, dr}{\int_{-\alpha}^{\alpha}\int_0^{c} r\, d\theta\, dr} = \frac{\frac{1}{3}c^3 \int_{-\alpha}^{\alpha} \cos\theta\, d\theta}{\frac{1}{2}c^2 \int_{-\alpha}^{\alpha} d\theta}$$

$$= \frac{2}{3}c \,.\, \frac{2\sin\alpha}{2\alpha} = \frac{2}{3}\left(\frac{\text{radius} \times \text{chord}}{\text{arc}}\right).$$

10. The area not being symmetrical, the formulæ to be used are

$$\bar{x} = \frac{\iint x\,dx\,dy}{\iint dx\,dy}; \quad \bar{y} = \frac{\iint y\,dx\,dy}{\iint dx\,dy}.$$

Integrating with respect to x, we have

$$\bar{x} = \frac{\frac{1}{2}\int x^2 dy}{\int x\,dy},$$

and from the equation to the circle

$$x^2 = a^2 - y^2;$$

$$\therefore\ \bar{x} = \frac{\frac{1}{2}\int_0^{a} (a^2 - y^2)\, dy}{\int_0^{a} (a^2 - y^2)^{\frac{1}{2}} dy} = \frac{\frac{1}{2}a^2 y - \frac{1}{6}y^3}{\frac{y}{2}(a^2 - y^2)^{\frac{1}{2}} + \frac{a^2}{2}\sin^{-1}\frac{y}{2}}$$

$$\text{between limits} = \frac{1}{3}a^3 \div \frac{a^2\pi}{4} = \frac{4a}{3\pi}.$$

$$\text{Also } \bar{y} = \frac{\frac{1}{2}\int y^2 dx}{\int y dx} = \frac{\frac{1}{2}\int (a^2 - x^2)\, dx}{\int (a^2 - x^2)^{\frac{1}{2}}\, dx} = \frac{\frac{1}{2}a^2 x - \frac{1}{6}x^3}{\frac{x}{2}(a^2 - x^2)^{\frac{1}{2}} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}}.$$

$$\text{between limits} = \frac{1}{3}a^3 \div \frac{a^2\pi}{4} = \frac{4a}{3\pi}.$$

12. $$\bar{x} = \frac{\iint x\,dx\,dy}{\iint dx\,dy} = \frac{\int_{ax}^{2m^{\frac{1}{2}}x^{\frac{1}{2}}}\int_0^x x\,dx\,dy}{\int_{ax}^{2m^{\frac{1}{2}}x^{\frac{1}{2}}}\int_0^x dx\,dy} = \frac{\int_0^x (2m^{\frac{1}{2}}x^{\frac{1}{2}} - ax)\, x\,dx}{\int_0^x (2m^{\frac{1}{2}}x^{\frac{1}{2}} - ax)\, dx}$$

$$= \frac{\int 2m^{\frac{1}{2}}x^{\frac{3}{2}}\, dx - \int ax^2 dx}{\int 2m^{\frac{1}{2}}x^{\frac{1}{2}}\, dx - \int ax\,dx} = \frac{\frac{4}{5}m^{\frac{1}{2}}x^{\frac{5}{2}} - \frac{1}{3}ax^3}{\frac{4}{3}m^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{1}{2}ax^2}.$$

Now $a^2x^2 = y^2 = 4mx$; $\therefore\ x = 0$ and $x = \frac{4m}{a^2}$, the limits of integration;

$$\therefore\ \bar{x} = \frac{64m^3}{15a^5} \div \frac{8m^2}{3a^3} = \frac{8m}{5a^2} \quad \text{......................... (1).}$$

$$\text{Again, } \bar{y} = \frac{\iint y\,dx\,dy}{\iint dx\,dy} = \frac{\frac{1}{2}\int_0^{\frac{4m}{a^2}} (4mx - a^2x^2)\, dx}{\int_0^{\frac{4m}{a^2}} (2m^{\frac{1}{2}}x^{\frac{1}{2}} - ax)\, dx}$$

$$= \frac{\frac{16m^3}{a^4} - \frac{32m^3}{3a^4}}{\frac{32m^2}{3a^3} - \frac{8m^2}{a^3}} = \frac{16m^3}{3a^4} \div \frac{8m^2}{3a^3} = \frac{2m}{a}.$$

18. For the centre of gravity, we have

$$\bar{x} = \frac{\int xy^2 dx}{\int y^2 dx} \quad \text{.............................(1),}$$

from the equation to the revolving circle

$$y^2 = a^2 - x^2,$$

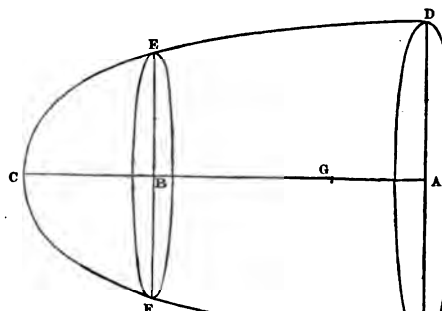
which being substituted in (1), we have

$$\bar{x} = \frac{\int_0^a (a^2 - x^2)\, x dx}{\int_0^a (a^2 - x^2)\, dx} = \frac{\frac{1}{2} a^2 x^2 - \frac{1}{4} x^4}{a^2 x - \frac{1}{3} x^3},$$

and between the limits $x = 0$, $x = a$;

$$\therefore \bar{x} = \frac{\frac{1}{2} a^4 - \frac{1}{4} a^4}{a^3 - \frac{1}{3} a^3} = \frac{a^4}{4} \div \frac{2a^3}{3} = \frac{3}{8} a.$$

22. Let $AD = a$, $BE = b$, $AB = h$; G the centre of gravity of frustum FD.



Also let $CA = p$, $CB = q$; $\therefore p - q = h$.

Then $p = \dfrac{a^2}{4m}$ and $q = \dfrac{b^2}{4m}$,

$$CG = \frac{\int xy^2 dx}{\int y^2 dx} = \frac{\int_q^p x^2 dx}{\int_q^p x dx} = \frac{2}{3}\left(\frac{p^3 - q^3}{p^2 - q^2}\right).$$

Hence $x = AG = p - \dfrac{2}{3}\left(\dfrac{p^2 + pq + q^2}{p + q}\right) = \dfrac{p^2 + pq - 2q^2}{3(p + q)}$

$$= \frac{p - q}{3} \times \frac{p + 2q}{p + q} = \frac{h}{3}\left(\frac{a^2 + 2b^2}{a^2 + b^2}\right).$$

24. Let α be the angle between the bounding radii of sector;

$$\text{then } \bar{x} = \frac{\int_0^\alpha \int_0^a r^3 \sin\theta\cos\theta\, d\theta dr}{\int_0^\alpha \int_0^a r^2 \sin\theta\, d\theta dr} = \frac{\frac{1}{4}a^4 \int_0^\alpha \sin\theta\cos\theta\, d\theta}{\frac{1}{3}a^3 \int_0^\alpha \sin\theta\, d\theta}$$

$$= \frac{3}{8}a \times \frac{\int_0^\alpha \sin 2\theta d\theta}{\int_0^\alpha \sin\theta\, d\theta} = \frac{3a}{16}\cdot\frac{1-\cos 2\alpha}{1-\cos\alpha} = \frac{3}{4}a\cos^2\frac{\alpha}{2}.$$

26.
$$\bar{x} = \frac{\iiint x dx dy dz}{\iiint dx dy dz} = \frac{\iint x dx\,.\,dy\,.\,z}{\iint z dx dy}$$

$$= \frac{\iint (r^2 - y^2 - x^2)^{\frac{1}{2}} x dx dy}{\iint z dx dy}.$$

But $\int (r^2 - y^2 - x^2)^{\frac{1}{2}} x dx = \frac{1}{3}(r^2 - y^2)^{\frac{3}{2}}$ taken between the limits $x = 0$ and $x = (r^2 - y^2)^{\frac{1}{2}}$,

and by the question we have

$$\iint z dx dy = \frac{1}{8} \text{ vol. of sphere} = \frac{1}{6}\pi r^3.$$

$$\text{Therefore } \bar{x} = \frac{2\int (r^2 - y^2)^{\frac{3}{2}} dy}{\pi r^3}.$$

But

$$\int (r^2 - y^2)^{\frac{3}{2}} dy = \left\{\frac{(r^2 - y^2)^{\frac{1}{2}}}{4} + \frac{3}{8}r^2\right\}.y(r^2 - y^2)^{\frac{1}{2}} + \frac{3r^4}{8}\sin^{-1}\frac{y}{r}$$

$$\text{from } y = 0 \text{ to } y = r = \frac{3}{16}\pi r^4;$$

$$\therefore \bar{x} = 2 \times \frac{3}{16}\pi r^4 \div \pi r^3 = \frac{3}{8}r.$$

Now the solid being symmetrical with respect to the three co-ordinate planes, we have

$$\bar{x} = \bar{y} = \bar{z} = \frac{3}{8}r.$$

28. The formulæ for the centre of gravity are

$$\bar{x} = \frac{\iiint x dx dy dz}{\iiint dx dy dz}, \quad \bar{y} = \frac{\iiint y dx dy dz}{\iiint dx dy dz}, \quad \bar{z} = \frac{\iiint z dx dy dz}{\iiint dx dy dz},$$

$$\iiint x dx dy dz = \iint xz dx dy, \text{ from } z = cx \text{ to } z = mx;$$

$$= (m-c) \iint x^2 dx dy = (m-c) \int y x^2 dx,$$

from $y = -\sqrt{2ax - x^2}$ to $y = +\sqrt{2ax - x^2}$;

$$= 2(m-c) \int_0^x x^2 (2ax - x^2)^{\frac{1}{2}} dx.$$

Similarly, we find

$$\iiint dx \,.\, dy \,.\, dz = 2(m-c) \int_0^x x (2ax - x^2)^{\frac{1}{2}} dx;$$

$$\therefore \ \bar{x} = \frac{\int_0^{2a} (2ax - x^2)^{\frac{1}{2}} x^2 dx}{\int_0^{2a} (2ax - x^2)^{\frac{1}{2}} x dx} = \frac{\frac{5\pi a^4}{8}}{\frac{\pi a^3}{2}} = \frac{5}{4} a,$$

$$\iiint y dx dy dz = \iint z y dy dx = \frac{1}{2} (m-c) \int y^2 x dx,$$

$$\text{from } y = -(2ax - x^2)^{\frac{1}{2}} \text{ to } y = +(2ax - x^2)^{\frac{1}{2}} = 0;$$

$$\therefore \ \bar{y} = 0,$$

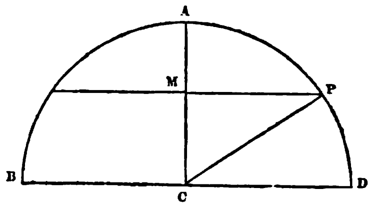
$$\iiint z dx dy dz = \iint \frac{1}{2} z^2 dy dx = \frac{1}{2} (m^2 - c^2) \int x^2 dx \,.\, y,$$

from $y = -(2ax - x^2)^{\frac{1}{2}}$ to $y = +(2ax - x^2)^{\frac{1}{2}}$

$$= (m^2 - c^2) \int x^2 (2ax - x^2)^{\frac{1}{2}} dx;$$

$$\therefore \ \bar{z} = \frac{(m^2 - c^2) \int_0^{2a} x^2 (2ax - x^2)^{\frac{1}{2}} dx}{2(m-c) \int_0^{2a} x (2ax - x^2)^{\frac{1}{2}} dx} = \frac{(m+c) \,.\, \frac{5\pi a^4}{8}}{2 \times \frac{\pi a^3}{2}} = \frac{5}{8} (m+c) a.$$

29. Let BAD be the semicircle, of which the centre is C, and radius a.



Let arc $AP = s$, $CM = x$;

$$\therefore\ x = a \cdot \cos\frac{s}{a};$$

$$\therefore \int x\,ds = a\int \cos\frac{s}{a}\,ds = a^2 \sin\frac{s}{a};$$

$$\therefore\ \bar{x} = \frac{\int_{-\frac{\pi a}{2}}^{\frac{\pi a}{2}} x\,ds}{\int_{-\frac{\pi a}{2}}^{\frac{\pi a}{2}} ds} = \frac{2a^2}{\pi a} = \frac{2a}{\pi}$$

32. Since $y = a \operatorname{vers}^{-1}\dfrac{x}{a} + (2ax - x^2)^{\frac{1}{2}}$;

$$\therefore\ \frac{dy}{dx} = \frac{2a - x}{(2ax - x^2)^{\frac{1}{2}}} = \left(\frac{2a - x}{x}\right)^{\frac{1}{2}}; \quad \therefore\ \frac{ds}{dx} = \left(\frac{2a}{x}\right)^{\frac{1}{2}}.$$

$$\bar{x} = \frac{\int x\,ds}{\int ds} = \frac{\int_0^{2a} (2ax)^{\frac{1}{2}}\,dx}{\int_0^{2a} (2ax^{-1})^{\frac{1}{2}}\,dx} = \frac{\frac{2}{3}(2a)^2}{2\,(2a)} = \frac{2}{3}a.$$

$$\text{Also } \bar{y} = \frac{\int y\,ds}{\int ds} = \frac{\int_0^{2a} y\left(\frac{2a}{x}\right)^{\frac{1}{2}} dx}{\int_0^{2a}\left(\frac{2a}{x}\right)^{\frac{1}{2}} dx} = \frac{2x^{\frac{1}{2}}y - 2\int x^{\frac{1}{2}}\frac{dy}{dx}\,dx}{\int x^{-\frac{1}{2}}\,dx}$$

from $x = 0$ to $x = 2a$

$$= \frac{(2a)^{\frac{1}{2}}\pi - 2\int_0^{2a}(2a - x)^{\frac{1}{2}}\,dx}{2\,(2a)^{\frac{1}{2}}} = \pi a - \frac{4a}{3}.$$

34. Since $y = \sin x$;

$$\therefore\ \frac{dy}{dx} = \cos x, \text{ and } \frac{ds}{dx} = (1 + \cos^2 x)^{\frac{1}{2}}.$$

$$\text{Hence } \bar{y} = \frac{\int y\,ds}{\int ds} = \frac{\int_0^\pi \sin x\,(1+\cos^2 x)^{\frac{1}{2}}\,dx}{\int_0^\pi (1+\cos^2 x)^{\frac{1}{2}}\,dx}.$$

If s denote the length of the curve between the limits $x=0$ and $x=\pi$, then will

$$\bar{y} = \frac{2^{\frac{1}{2}} + \log(2^{\frac{1}{2}}+1)}{s}.$$

35. From the equation to the generating curve we have

$$ds^2 = dx^2 + dy^2 = \frac{a^2}{(2ax - x^2)} \cdot dx^2 = \frac{a^2 dx^2}{y^2},$$

$$\text{or } y\,ds = a\,dx;$$

$$\therefore\ \bar{x} = \frac{\int xy\,ds}{\int y\,ds} = \frac{\int_0^c ax\,dx}{\int_0^c a\,dx} = \frac{\frac{1}{2}c^2}{c} = \frac{1}{2}c.$$

36. Let $y = ax$ be the equation to the generating line; then

$$dy = a\,dx,\quad dy^2 + dx^2 = (a^2+1)\,dx^2;$$

$$\therefore\ \int_0^a xy\,ds = \int_0^a a\,(a^2+1)^{\frac{1}{2}} x^2\,dx = \frac{a^4(a^2+1)^{\frac{1}{2}}}{3},$$

$$\text{and } \int_0^a y\,ds = \int_0^a a\,(a^2+1)^{\frac{1}{2}} x\,dx = \frac{a^3(a^2+1)^{\frac{1}{2}}}{2};$$

$$\therefore\ \bar{x} = \frac{\int_0^a xy\,ds}{\int_0^a y\,ds} = \frac{a^4(a^2+1)^{\frac{1}{2}}}{3} \div \frac{a^3(a^2+1)^{\frac{1}{2}}}{2} = \frac{2a}{3}.$$

38. Since $y = a\,\text{vers}^{-1}\dfrac{x}{a} + (2ax - x^2)^{\frac{1}{2}}$;

$$\therefore\ \frac{dy}{dx} = \left(\frac{2a-x}{x}\right)^{\frac{1}{2}},\ \text{and}\ \therefore\ \frac{ds}{dx} = \left(\frac{2a}{x}\right)^{\frac{1}{2}},$$

$$\int xy\,ds = \int y\,(2ax)^{\frac{1}{2}}\,dx = (2a)^{\frac{1}{2}}\left\{\frac{2}{3}x^{\frac{3}{2}}y - \frac{2}{3}\int x\,(2a-x)^{\frac{1}{2}}\,dx\right\};$$

$$\therefore \int_0^{2a} xyds = (2a)^{\frac{1}{2}} \left\{ \frac{2}{3} (2a)^{\frac{3}{2}} (a\pi) - \frac{2}{3} \left(\frac{4}{15} \right) (2a)^{\frac{5}{2}} \right\} = \frac{(2a)^3}{3} \left(\pi - \frac{8}{15} \right).$$

$$\int yds = \int_0^{2a} y \left(\frac{2a}{x} \right)^{\frac{1}{2}} dx = (2a)^{\frac{1}{2}} \left\{ 2x^{\frac{1}{2}} y - 2 \int_0^{2a} (2a - x)^{\frac{1}{2}} dx \right\}$$

$$= (2a^{\frac{1}{2}} \left\{ 2 (2a)^{\frac{1}{2}} (a\pi) - \frac{4}{3} (2a)^{\frac{3}{2}} \right\} = (2a)^2 \left(\pi - \frac{4}{3} \right);$$

$$\therefore \bar{x} = \frac{\int xyds}{\int yds} = \frac{2a}{3} \left\{ \frac{\pi - \frac{8}{15}}{\pi - \frac{4}{3}} \right\} = \frac{2a}{15} \left\{ \frac{15\pi - 8}{3\pi - 4} \right\}.$$

39. Let AB be the given physical line, and C the given point; $CA = a$, $CB = b$, $CP = x$.

The density of the elementary portion PQ varies inversely as the square of its distance from C;

$$\therefore dm = \frac{\rho}{x^2} dx,$$

ρ being the density at the distance of unity.

$$\text{Then } \int xdm = \rho \int_a^b x \frac{dx}{x^2} = \rho \log \frac{b}{a},$$

$$\text{and } \int dm = \rho \int_a^b \frac{dx}{x^2} = \rho \left(\frac{1}{a} - \frac{1}{b} \right);$$

$$\therefore \bar{x} = \frac{\int xdm}{\int dm} = \log \frac{b}{a} \div \left(\frac{1}{a} - \frac{1}{b} \right) = \frac{ab}{b-a} \log \frac{b}{a}.$$

42. The equation to the generating circle is

$$y^2 = a^2 - x^2,$$

and for the centre of gravity we have

$$\bar{x} = \frac{\int_0^a \rho xy^2 dx}{\int_0^a \rho y^2 dx};$$

$$\text{but } \rho \propto y^2; \quad \therefore \bar{x} = \frac{\int_0^a (a^2 - x^2)^2 \, x dx}{\int_0^a (a^2 - x^2)^2 \, dx}.$$

$$\text{But } \int (a^2 - x^2)^2 \, x dx = -\frac{1}{6}(a^2 - x^2)^3 = \frac{1}{6} a^6;$$

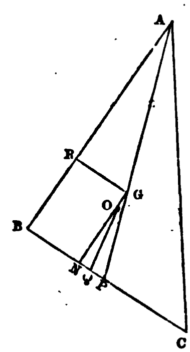
from $x = 0$ to $x = a$;

$$\text{and } \int (a^2 - x^2)^2 \, dx = a^4 x - \frac{2}{3} a^2 x^3 + \frac{1}{5} x^5 = \frac{8}{15} a^5;$$

from $x = 0$ to $x = a$;

$$\therefore \bar{x} = \frac{a^6}{6} \div \frac{8a^5}{15} = \frac{5a}{16}.$$

Ex. 7.



1. Let ABC be the triangle, G its centre of gravity, and O the centre of the inscribed circle. Join AG and produce it to meet the base BC in P. From G draw a perpendicular to BC, this perpendicular will pass through O; if not, let it fall otherwise, as GQ. From O draw ON perpendicular to BC; and from G draw GR perpendicular to AB and therefore parallel to BC, for the triangle ABC is right-angled at B.

$$\text{Then } GQ = \frac{1}{3} AB,$$

$$\text{and } GR = BQ = \frac{2}{3} BP = \frac{1}{3} BC = 1,$$

$$\text{and by Trigonometry } ON = BN = \frac{2 \times \text{area of } \Delta}{\text{perimeter}} = 1,$$

$$\text{or } BN = BQ.$$

Therefore the point Q coincides with N; that is, the perpendicular from G on BC will pass through O, or GON is perpendicular to BC. But by the property of the centre of

Also $\bar{y} = \dfrac{\frac{1}{2}\int y^2dx}{\int ydx} = \dfrac{\frac{1}{2}\int (a^2 - x^2)\,dx}{\int (a^2 - x^2)^{\frac{1}{2}}\,dx} = \dfrac{\frac{1}{2}a^2x - \frac{1}{6}x^3}{\frac{x}{2}(a^2 - x^2)^{\frac{1}{2}} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}}$

$$\text{between limits} = \frac{1}{3}a^3 \div \frac{a^2\pi}{4} = \frac{4a}{3\pi}.$$

12. $\bar{x} = \dfrac{\iint xdxdy}{\iint dxdy} = \dfrac{\int_{ax}^{2m^{\frac{1}{2}}x^{\frac{1}{2}}}\int_0^x xdxdy}{\int_{ax}^{2m^{\frac{1}{2}}x^{\frac{1}{2}}}\int_0^x dxdy} = \dfrac{\int_0^x (2m^{\frac{1}{2}}x^{\frac{1}{2}} - ax)\,xdx}{\int_0^x (2m^{\frac{1}{2}}x^{\frac{1}{2}} - ax)\,dx}$

$$= \frac{\int 2m^{\frac{1}{2}}x^{\frac{3}{2}}\,dx - \int ax^2dx}{\int 2m^{\frac{1}{2}}x^{\frac{1}{2}}\,dx - \int axdx} = \frac{\frac{4}{5}m^{\frac{1}{2}}x^{\frac{5}{2}} - \frac{1}{3}ax^3}{\frac{4}{3}m^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{1}{2}ax^2}.$$

Now $a^2x^2 = y^2 = 4mx$; $\therefore\ x = 0$ and $x = \frac{4m}{a^2}$, the limits of integration;

$$\therefore\ \bar{x} = \frac{64m^3}{15a^5} \div \frac{8m^2}{3a^3} = \frac{8m}{5a^2} \quad \text{......................... (1).}$$

$$\text{Again, } \bar{y} = \frac{\iint ydxdy}{\iint dxdy} = \frac{\frac{1}{2}\int_0^{\frac{4m}{a^2}}(4mx - a^2x^2)\,dx}{\int_0^{\frac{4m}{a^2}}(2m^{\frac{1}{2}}x^{\frac{1}{2}} - ax)\,dx}$$

$$= \frac{\frac{16m^3}{a^4} - \frac{32m^3}{3a^4}}{\frac{32m^2}{3a^3} - \frac{8m^2}{a^3}} = \frac{16m^3}{3a^4} \div \frac{8m^2}{3a^3} = \frac{2m}{a}.$$

18. For the centre of gravity, we have

$$\bar{x} = \frac{\int xy^2dx}{\int y^2dx} \quad \text{.........................(1),}$$

and from the equation to the revolving circle

$$y^2 = a^2 - x^2,$$

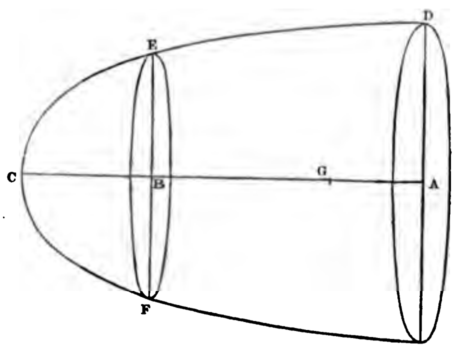
which being substituted in (1), we have

$$\bar{x} = \frac{\int_0^a (a^2 - x^2)\, x dx}{\int_0^a (a^2 - x^2)\, dx} = \frac{\frac{1}{2} a^2 x^2 - \frac{1}{4} x^4}{a^2 x - \frac{1}{3} x^3},$$

and between the limits $x = 0$, $x = a$;

$$\therefore \bar{x} = \frac{\frac{1}{2} a^4 - \frac{1}{4} a^4}{a^3 - \frac{1}{3} a^3} = \frac{a^4}{4} \div \frac{2a^3}{3} = \frac{3}{8} a.$$

22. Let $AD = a$, $BE = b$, $AB = h$; G the centre of gravity of frustum FD.



Also let $CA = p$, $CB = q$; $\therefore p - q = h$.

Then $p = \frac{a^2}{4m}$ and $q = \frac{b^2}{4m}$,

$$CG = \frac{\int xy^2 dx}{\int y^2 dx} = \frac{\int_q^p x^2 dx}{\int_q^p x dx} = \frac{2}{3}\left(\frac{p^3 - q^3}{p^2 - q^2}\right).$$

Hence $x = AG = p - \frac{2}{3}\left(\frac{p^2 + pq + q^2}{p + q}\right) = \frac{p^2 + pq - 2q^2}{3(p+q)}$

$$= \frac{p-q}{3} \times \frac{p + 2q}{p + q} = \frac{h}{3}\left(\frac{a^2 + 2b^2}{a^2 + b^2}\right).$$

24. Let a be the angle between the bounding radii of sector;

$$\text{then } \bar{x} = \frac{\int_0^a \int_0^a r^3 \sin\theta \cos\theta \, d\theta dr}{\int_0^a \int_0^a r^2 \sin\theta \, d\theta dr} = \frac{\frac{1}{4} a^4 \int_0^a \sin\theta \cos\theta \, d\theta}{\frac{1}{3} a^3 \cdot \int_0^a \sin\theta \, d\theta}$$

$$= \frac{3}{8} a \times \frac{\int_0^a \sin 2\theta d\theta}{\int_0^a \sin\theta \, d\theta} = \frac{3a}{16} \cdot \frac{1 - \cos 2\alpha}{1 - \cos\alpha} = \frac{3}{4} a \cos^2 \frac{\alpha}{2}.$$

26. $$\bar{x} = \frac{\iiint x dx dy dz}{\iiint dx dy dz} = \frac{\iint x dx \, . \, dy \, . \, z}{\iint z dx dy}$$

$$= \frac{\iint (r^2 - y^2 - x^2)^{\frac{1}{2}} x dx dy}{\iint z dx dy}.$$

But $\int (r^2 - y^2 - x^2)^{\frac{1}{2}} x dx = \frac{1}{3} (r^2 - y^2)^{\frac{3}{2}}$ taken between the limits $x = 0$ and $x = (r^2 - y^2)^{\frac{1}{2}}$,

and by the question we have

$$\iint z dx dy = \frac{1}{8} \text{ vol. of sphere} = \frac{1}{6} \pi r^3.$$

$$\text{Therefore } \bar{x} = \frac{2 \int (r^2 - y^2)^{\frac{3}{2}} dy}{\pi r^3}.$$

But

$$\int (r^2 - y^2)^{\frac{3}{2}} dy = \left\{ \frac{(r^2 - y^2)^{\frac{1}{2}}}{4} + \frac{3}{8} r^2 \right\} . y (r^2 - y^2)^{\frac{1}{2}} + \frac{3r^4}{8} \sin^{-1} \frac{y}{r}$$

$$\text{from } y = 0 \text{ to } y = r = \frac{3}{16} \pi r^4;$$

$$\therefore \bar{x} = 2 \times \frac{3}{16} \pi r^4 \div \pi r^3 = \frac{3}{8} r.$$

Now the solid being symmetrical with respect to the three co-ordinate planes, we have

$$\bar{x} = \bar{y} = \bar{z} = \frac{3}{8} r.$$

28. The formulæ for the centre of gravity are

$$\bar{x}=\frac{\iiint xdxdydz}{\iiint dxdydz},\quad \bar{y}=\frac{\iiint ydxdydz}{\iiint dxdydz},\quad \bar{z}=\frac{\iiint zdxdydz}{\iiint dxdydz},$$

$$\iiint xdxdydz=\iint xzdxdy, \text{ from } z=cx \text{ to } z=mx;$$

$$=(m-c)\iint x^2dxdy=(m-c)\int yx^2dx,$$

from $y=-\sqrt{2ax-x^2}$ to $y=+\sqrt{2ax-x^2}$;

$$=2(m-c)\int_0^x x^2(2ax-x^2)^{\frac{1}{2}}dx.$$

Similarly, we find

$$\iiint dx.dy.dz=2(m-c)\int_0^x x(2ax-x^2)^{\frac{1}{2}}dx;$$

$$\therefore\ \bar{x}=\frac{\int_0^{2a}(2ax-x^2)^{\frac{1}{2}}x^2\,dx}{\int_0^{2a}(2ax-x^2)^{\frac{1}{2}}xdx}=\frac{\frac{5\pi a^4}{8}}{\frac{\pi a^3}{2}}=\frac{5}{4}a,$$

$$\iiint ydxdydz=\iint zydydx=\frac{1}{2}(m-c)\int y^2xdx,$$

from $y=-(2ax-x^2)^{\frac{1}{2}}$ to $y=+(2ax-x^2)^{\frac{1}{2}}=0$;

$$\therefore\ \bar{y}=0,$$

$$\iiint zdxdydz=\iint\frac{1}{2}z^2dydx=\frac{1}{2}(m^2-c^2)\int x^2dx.y,$$

from $y=-(2ax-x^2)^{\frac{1}{2}}$ to $y=+(2ax-x^2)^{\frac{1}{2}}$

$$=(m^2-c^2)\int x^2(2ax-x^2)^{\frac{1}{2}}dx;$$

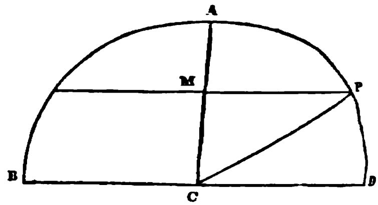
$$\therefore\ \bar{z}=\frac{(m^2-c^2)\int_0^{2a}x^2(2ax-x^2)^{\frac{1}{2}}dx}{2(m-c)\int_0^{2a}x(2ax-x^2)^{\frac{1}{2}}dx}=\frac{(m+c).\frac{5\pi a^4}{8}}{2\times\frac{\pi a^3}{2}}=\frac{5}{8}(m+c)a.$$

29. Let BAD be the semicircle, of which the centre is C, and radius a.

Let arc $AP = s$, $CM = x$;

$$\therefore\ x = a \cdot \cos \frac{s}{a};$$

$$\therefore\ \int x\,ds = a \int \cos \frac{s}{a}\,ds = a^2 \sin \frac{s}{a};$$



$$\therefore\ \bar{x} = \frac{\int_{-\frac{\pi a}{2}}^{\frac{\pi a}{2}} x\,ds}{\int_{-\frac{\pi a}{2}}^{\frac{\pi a}{2}} ds} = \frac{2a^2}{\pi a} = \frac{2a}{\pi}$$

32. Since $y = a \operatorname{vers}^{-1} \frac{x}{a} + (2ax - x^2)^{\frac{1}{2}}$;

$$\therefore\ \frac{dy}{dx} = \frac{2a - x}{(2ax - x^2)^{\frac{1}{2}}} = \left(\frac{2a - x}{x}\right)^{\frac{1}{2}};\quad \therefore\ \frac{ds}{dx} = \left(\frac{2a}{x}\right)^{\frac{1}{2}}.$$

$$\bar{x} = \frac{\int x\,ds}{\int ds} = \frac{\int_0^{2a} (2ax)^{\frac{1}{2}}\,dx}{\int_0^{2a} (2ax^{-1})^{\frac{1}{2}}\,dx} = \frac{\frac{2}{3}(2a)^2}{2\,(2a)} = \frac{2}{3}a.$$

$$\text{Also } \bar{y} = \frac{\int y\,ds}{\int ds} = \frac{\int_0^{2a} y \left(\frac{2a}{x}\right)^{\frac{1}{2}} dx}{\int_0^{2a} \left(\frac{2a}{x}\right)^{\frac{1}{2}} dx} = \frac{2x^{\frac{1}{2}}\,y - 2\int x^{\frac{1}{2}} \frac{dy}{dx}\,dx}{\int x^{-1}\,dx}$$

from $x = 0$ to $x = 2a$

$$= \frac{(2a)^{\frac{1}{2}}\,\pi - 2\int_0^{2a} (2a - x)^{\frac{1}{2}}\,dx}{2\,(2a)^{\frac{1}{2}}} = \pi a - \frac{4a}{3}.$$

34. Since $y = \sin x$;

$$\therefore\ \frac{dy}{dx} = \cos x, \text{ and } \frac{ds}{dx} = (1 + \cos^2 x)^{\frac{1}{2}}.$$

$$\text{Hence } \bar{y} = \frac{\int y ds}{\int ds} = \frac{\int_0^\pi \sin x\,(1+\cos^2 x)^{\frac{1}{2}}\,dx}{\int_0^\pi (1+\cos^2 x)^{\frac{1}{2}}\,dx}.$$

If s denote the length of the curve between the limits $x = 0$ and $x = \pi$, then will

$$\bar{y} = \frac{2^{\frac{1}{2}} + \log(2^{\frac{1}{2}} + 1)}{s}.$$

35. From the equation to the generating curve we have

$$ds^2 = dx^2 + dy^2 = \frac{a^2}{(2ax - x^2)} \cdot dx^2 = \frac{a^2 dx^2}{y^2},$$

$$\text{or } yds = adx;$$

$$\therefore \bar{x} = \frac{\int xyds}{\int yds} = \frac{\int_0^c axdx}{\int_0^c adx} = \frac{\frac{1}{2}c^2}{c} = \frac{1}{2}c.$$

36. Let $y = ax$ be the equation to the generating line; then

$$dy = adx,\quad dy^2 + dx^2 = (a^2+1)\,dx^2;$$

$$\therefore \int_0^a xyds = \int_0^a a\,(a^2+1)^{\frac{1}{2}} x^2\,dx = \frac{a^4\,(a^2+1)^{\frac{1}{2}}}{3},$$

$$\text{and } \int_0^a yds = \int_0^a a\,(a^2+1)^{\frac{1}{2}}\,xdx = \frac{a^3\,(a^2+1)^{\frac{1}{2}}}{2};$$

$$\therefore \bar{x} = \frac{\int_0^a xyds}{\int_0^a yds} = \frac{a^4\,(a^2+1)^{\frac{1}{2}}}{3} \div \frac{a^3\,(a^2+1)^{\frac{1}{2}}}{2} = \frac{2a}{3}.$$

38. Since $y = a \operatorname{vers}^{-1}\frac{x}{a} + (2ax - x^2)^{\frac{1}{2}}$;

$$\therefore \frac{dy}{dx} = \left(\frac{2a-x}{x}\right)^{\frac{1}{2}},\ \text{and}\ \therefore \frac{ds}{dx} = \left(\frac{2a}{x}\right)^{\frac{1}{2}},$$

$$\int xyds = \int y\,(2ax)^{\frac{1}{2}}\,dx = (2a)^{\frac{1}{2}}\left\{\frac{2}{3}x^{\frac{3}{2}}y - \frac{2}{3}\int x\,(2a-x)^{\frac{1}{2}}\,dx\right\};$$

$$\therefore \frac{2}{3} hk \times 2\pi\bar{y} = \frac{1}{2} \pi k^2 h;$$

$$\therefore \bar{y} = \frac{3}{8} k = \frac{3}{8} \times \text{extreme ordinate } BC.$$

10. Let c denote the distance of the centre of the ellipse from the axis of revolution; then the length of the path described by the centre of gravity $= \pi c$; and the area of the ellipse $= \pi ab$ (a, b, being the semiaxes); therefore

$$\text{volume of ring} = \pi ab \times \pi c = \pi^2 abc.$$

MACHINES.

Lever.

Ex. 10.

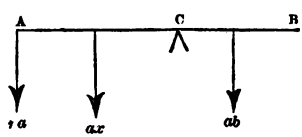
1. The weight of the lever acts at its middle point, and, consequently, at the distance of half a foot from the fulcrum, and on the same side of it as the given weight.

Let $x =$ the weight required.

$$\text{Then } x \times 1 = \frac{1}{2} \times 4 + 2 \times 10;$$

$$\therefore x = 22 \text{ lbs.}$$

5. Let AB be the rod, and C the fulcrum upon which it rests. Then if $AC = x$, its weight will be ax, and will act at its middle point. Therefore



$$nax + \frac{ax^2}{2} = \frac{ab^2}{2};$$

$$\therefore x^2 + 2nx = b^2,$$

$$\text{whence } x = (b^2 + n^2)^{\frac{1}{2}} - n;$$

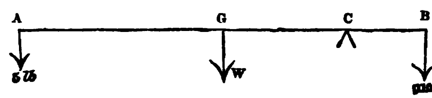
$$\therefore AB = (b - n) + (b^2 + n^2)^{\frac{1}{2}} \text{ the length of the rod.}$$

8. Let $x =$ the dist. of the fulcrum, or centre of gravity of the system, from the end where the smallest body is attached.

$$\text{Then } x = \frac{2P\times a + 3P\times 2a + 4P\times 3a + \ldots + (n+1)\,P\times na}{P + 2P + 3P + \ldots\ldots + (n+1)\,P}$$

$$= a\,.\left\{\frac{\frac{1}{3}n\,(n+1)\,(n+2)}{\frac{1}{2}\,(n+1)\,(n+2)}\right\} = \frac{2}{3}\,na.$$

11. Let W be the weight of the beam, which being uniform will have its centre of gravity at G, the middle point of AB.

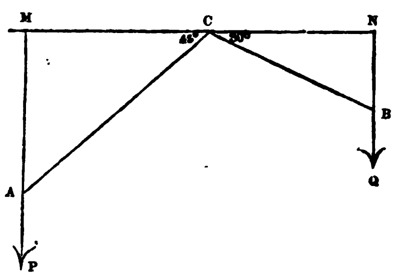


Taking moments about C, we have

$$W\times CG + 5\times CA = 110\times CB;$$

$$\therefore\; W = \frac{110\times 2 - 5\times 16}{7} = \frac{140}{7} = 20\,\text{lbs.}$$

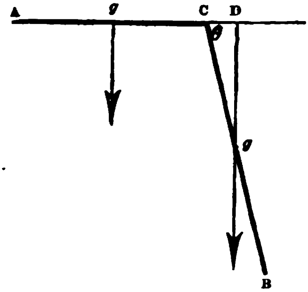
14. In the lever ACB let $CA = 18$ in., $CB = 12$.



Then, taking moments about C, $P\times CM = Q\times CN$;

$$\therefore\; P : Q = CB\cos 30^\circ : CA\cos 45^\circ = 2\times\frac{1}{2}\sqrt{3} : 3\times\frac{1}{2}\sqrt{2}$$

$$= \sqrt{2} : \sqrt{3}.$$



17. Let ACB be the lever, and let the angle $BCD = \theta$. Let $2x$ = length of shorter arm in first instance, and $2y$ = length of longer arm. Then, on the first supposition, taking the moments about C, we have

$$x \times Cg = y \times CD,$$

$$\text{or } x^2 = y^2 \cos\theta,$$

and on the second supposition, we have

$$y^2 = 4x^2 \cos\theta;$$

$$\therefore \frac{x^2}{\cos\theta} = 4x^2 \cos\theta; \quad \therefore \cos^2\theta = \frac{1}{4},$$

$$\cos\theta = \frac{1}{2}, \quad \theta = 60°;$$

$\therefore$ the angle between the arms $= 120°$,

$$\text{also } x^2 : y^2 = \frac{1}{2} : 1;$$

$$\therefore x : y = 1 : \sqrt{2}.$$

20. When there is equilibrium we must have

$$P : W = BC \sin\beta : AC \sin\alpha,$$

α, β being the inclination of the forces to the arms AC, BC respectively.

But, by the question, we have $BC = AC$, and CE being parallel to the forces

$$BCE = \beta, \text{ and } ACE = \alpha;$$

$$\therefore P : W :: \sin BCE : \sin ACE.$$

Therefore, componendo et dividendo,

$$P + W : P - W = \sin BCE + \sin ACE : \sin BCE - \sin ACE,$$

$$\text{or } \frac{P+W}{P-W} = \frac{\tan\frac{1}{2}(BCE + ACE)}{\tan\frac{1}{2}(BCE - ACE)},$$

$$\text{but } BCE + ACE = ACB,$$

$$\text{and } BCE - ACE = 2DCE;$$

$$\therefore P + W : P - W = \tan \frac{1}{2}(ACB) : \tan DCE.$$

23. Let x = the true weight; a, b, the arms of the beam upon which the weights 9 lbs. and 4 lbs. respectively, are suspended. Then

$$x : 9 = a : b \text{.......................} (1),$$

$$\text{and } x : 4 = b : a \text{.......................} (2);$$

$$\therefore x^2 = 36; \quad \therefore x = 6 \text{ lbs.}$$

Let z be the number of inches in the longer arm.

$$\text{Then } 6 : 4 = z : 45 - z;$$

$$\therefore 6 : 10 = z : 45;$$

$$\therefore z = \frac{6 \times 45}{10} = 27 \text{ in.} = 2 \text{ ft. } 3 \text{ in.}$$

The arms are therefore 2 ft. 3 in. and 1 ft. 6 in.

27. When the weight w is put into the scale of the shorter arm, let p be the weight of the goods put into the scale of the longer arm.

$$\text{Then } p(m+1) = w.m, \quad \therefore p = \frac{m}{m+1} w.$$

Also let q from the shorter arm balance w from the longer, then we have

$$q \cdot m = w(m+1); \quad \therefore q = \frac{m+1}{m} w.$$

In the latter case the seller *loses* $q - w = \dfrac{w}{m}$.

...... former .... *gains* $w - p = \dfrac{w}{m+1}$.

Hence, in any such pair of sales,

the seller's *loss* $= \dfrac{w}{m} - \dfrac{w}{m+1} = \dfrac{w}{m(m+1)}$;

$\therefore$ in one hundred sales his *loss* $= \dfrac{50w}{m(m+1)}$,

i.e. the seller's *loss* $= \dfrac{50}{m(m+1)}$ per cent.

30. Let G be the centre of gravity of the beam, at which point its weight $W = 3$ lbs. or 48 oz. is supposed to act. When the fulcrum is at C, let P at A balance W at G.

$$AC = x, \quad AG = 18 \text{ in.}; \quad \therefore CG = 18 - x.$$

$$\text{Then } Px = W(18 - x); \quad \therefore x = \frac{48 \times 18}{P + 48}.$$

Let $x_1, x_2, x_3, \ldots\ldots$ be the values of x when the values of P are 4, 8, 12, &c. *oz.* respectively.

$$\text{Then } x_1 = \frac{48 \times 18}{4 + 48} = \frac{12 \times 18}{13} = 16\tfrac{8}{13} \text{ inches},$$

$$x_2 = \frac{48 \times 18}{8 + 48} = \frac{6 \times 18}{7} = 15\tfrac{3}{7} \ldots\ldots$$

$$x_3 = \frac{48 \times 18}{12 + 48} = \frac{4 \times 18}{5} = 14\tfrac{2}{5} \ldots\ldots$$

&c. &c. &c.

Wheel and Axle.

Ex. 11.

1. $P : W ::$ rad. of axle : rad. of wheel;

$$\therefore 10 : 555 = \text{rad. of axle} : 2 \text{ yds.};$$

$$\therefore \text{rad. of axle} = \frac{720 \text{ in.}}{555} = 1\tfrac{11}{37} \text{ in.}$$

4. Let x = the weight required. Then

rad. of larger wheel × 48 lbs. + rad. of smaller wheel × 50 lbs.

$$= \text{rad. of axle} \times x \text{ lbs.},$$

$$\text{or } 2\tfrac{1}{2} \times 48 + 2 \times 50 = \frac{5}{6}x; \quad \text{or } \frac{5}{6}x = 220;$$

$$\therefore x = 264 \text{ lbs.}$$

PULLEY.

Ex. 12.

1. By the formula we have

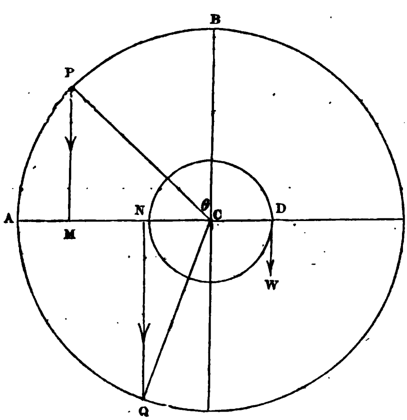
$$2P \cos \alpha = W,$$

2α being the inclination of the strings; and by the question,

$$2P \cos \alpha = P, \text{ or } \cos \alpha = \frac{1}{2} = \cos 60°;$$

$\therefore$ the angle between the strings is 120°.

7. Let $P = 7$ lbs. and $Q = 5$ lbs., acting as represented in the figure, support the weight W at D, $AC = a$, $CD = b$, $\angle PCB = \theta$.



Taking moments about C,

$$P \times CM + Q \times CN = W \times CD;$$

$$\therefore 7a \sin\theta + 5a \sin(60° - \theta) = Wb.$$

Hence by the question,

$$7 \sin\theta + 5 \sin(60° - \theta) = \frac{Wb}{a} = \text{maximum}.$$

Differentiating with regard to θ, we obtain

$$7 \cos\theta - 5 \cos(60° - \theta) = 0,$$

$$\text{hence } \tan\theta = \frac{3\sqrt{3}}{5} = \tan 46° \, 6' \, 7''\cdot 6.$$

8. Let t_1 be the tension of the string passing round the lowest pulley w,

t_2 be the tension of the string passing round the second pulley $w' = 2w$,

t_3 be the tension of the string passing round the third pulley $w'' = 2^2 w$.

. . . .

Then $t_1 = \dfrac{W+w}{2}$,

$$t_2 = \frac{1}{2}\left(\frac{W+w}{2} + w'\right) = \frac{W+w}{2^2} + \frac{w'}{2},$$

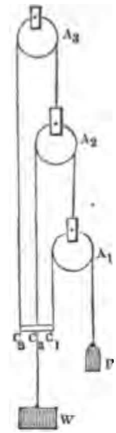
$$t_3 = \frac{1}{2}\left(\frac{W+w}{2^2} + \frac{w'}{2} + w''\right) = \frac{W+w}{2^3} + \frac{w'}{2^2} + \frac{w''}{2},$$

$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$t_n = \frac{W+w}{2^n} + \frac{w'}{2^{n-1}} + \frac{w''}{2^{n-2}} + \ldots\ldots + \frac{w^{(n-1)}}{2} = P;$$

$$\therefore W = 2^n P - w(1 + 2 \times 2 + 2^2 \times 2^2 + 2^3 \times 2^3 + \ldots + 2^{n-1} \times 2^{n-1})$$

$$= 2^n P - \frac{1}{3}(4^n - 1)\, w.$$



11. Let w be the weight of each pulley.

$t_1, t_2, t_3 \ldots\ldots t_8$ the tensions of the 1st, 2nd, 3rd......8th strings respectively, or the portions of the weight supported at $C_1, C_2, C_3 \ldots\ldots$

Then

$$t_1 = P \qquad = P + (1-1)\,w,$$

$$t_2 = 2P + w \qquad = 2P + (2-1)\,w,$$

$$t_3 = 2\,(2P+w) + w = 2^2P + (2^2-1)\,w,$$

$$t_4 = 2\,(2^2P + 3w) + w = 2^3P + (2^3-1)\,w,$$

$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$t_8 = \qquad = 2^7P + (2^7-1)\,w.$$

Hence

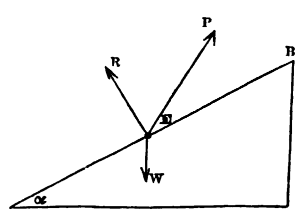
$$W = t_1 + t_2 + \ldots\ldots + t_8 = (P+w)\,(1 + 2 + 2^2 + \ldots\ldots + 2^7) - 8w$$

$$= (2^8-1)\,P + (2^8-9)\,w.$$

If $P=0$, $w : W = 1 : 2^8 - 9 = 1 : 247$.

Inclined Plane.

Ex. 13.



3. Let $P = 5m$,

$R = 4m$,

$W = 7m$;

resolve these forces parallel and perpendicular to AB.

Then

$$\left.\begin{aligned} P\cos\epsilon &= W\sin\alpha, \\ R + P\sin\epsilon &= W\cos\alpha, \end{aligned}\right\}\text{ to find } \epsilon \text{ and } \alpha.$$

Square both equations and add

$$R^2 + 2RP\sin\epsilon + P^2(\cos^2\epsilon + \sin\epsilon) = W^2(\sin^2\alpha + \cos^2\alpha);$$

$$\therefore \sin\epsilon = \frac{W^2 - P^2 - R^2}{2RP} = \frac{7^2 - 5^2 - 4^2}{2\times 5\times 4} = \frac{8}{8\times 5} = \cdot 2 \sin 11^\circ\ 32'.$$

$$\text{Also}\quad \cos\alpha = \frac{R + P\sin\epsilon}{W} = \frac{4+1}{7} = \cos 44^\circ\ 25'.$$

We may also find α from the equation $\sin\alpha = \frac{5}{7}\cos\epsilon$.

5. Let θ be the inclination of the plane to the horizon,

T be the tension of the string parallel to the plane.

Then $W : T = 1 : \sin\theta$, by the property of the inclined plane,

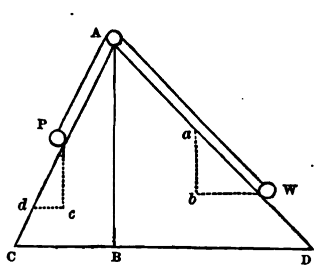
and $T : P = R : r$, ... wheel and axle;

$$\therefore W : P = R : r\sin\theta;$$

$$\therefore \theta = \sin^{-1}\left(\frac{PR}{Wr}\right).$$

7. Referring to the figure in the margin; suppose the weight drawn up the plane AD by the power P acting over the pulley at A; then

$$P : W = AC : AD,$$



and while P descends through a space Wa, W ascends upon the plane through the same space; but the space actually described by W in this time in the direction of gravity is ba; and the space described by P in the direction of gravity is Pc; therefore, when the velocity of the power and weight are estimated in the direction in which they respectively act, we have

$$\text{virt. vel. of } P : \text{virt. vel. of } W = Pc : ba = \sin ACB : \sin ADB$$
$$= AD : AC = W : P.$$

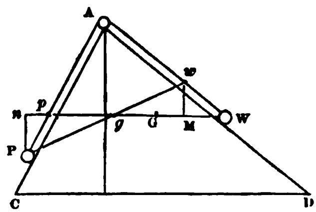


fig. 2.

Again, referring to fig. (2), let p. W be the position of the power and weight when in the same horizontal line. Let p descend to P, and W ascend to w; therefore $Pp = Ww$. Join wP meeting Wp in g; and draw wm, Pn perpendicular to pW. Now by similar triangles we have

$$wg : Pg = wm : Pn = AC : AD = P : W;$$

therefore g is the centre of gravity of P and w. Hence the centre of gravity has moved in the horizontal line Gg; and this is true whatever be the space described.

Screw.

Ex. 14.

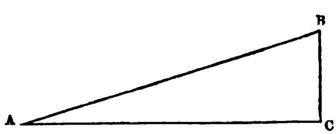
1. The formula is

$$P : W = \text{dist. between two threads : circum. described by } P;$$

$$\therefore P : W = 2 : 2\pi \times 20 = 1 : 20\pi$$

$$= 1 : 20 \times 3{\cdot}1416 = 1 : 62{\cdot}83.$$

5. If AC = circumference of a transverse section of the screw cylinder,



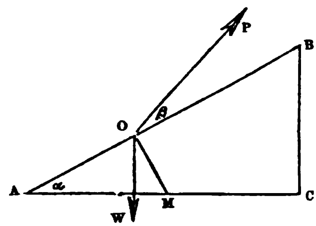
then $d = CB = AC \tan 30° = 2\pi \times 9 \times \dfrac{1}{\sqrt{3}} = 2\pi \times 3\sqrt{3}$.

$$\text{Hence } P : 15 \times 112 = 2\pi \times 3\sqrt{3} : 2\pi \times 48;$$

$$\therefore P = \frac{15 \times 112\sqrt{3}}{16} = 105\sqrt{3} = 181{\cdot}865 \text{ lbs.}$$

FRICTION.

Ex. 15.



1. Let the diagram be as in the margin.

Resolving the forces P and W in directions parallel and perpendicular to AB, we obtain the whole pressure perpendicular to the plane

$$= W\cos\alpha - P\sin\beta.$$

Therefore friction $= \mu(W\cos\alpha - P\sin\beta)$.

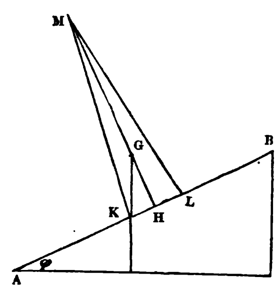
Therefore for the pressures in directions AB and BA respectively, we have

$$P\cos\beta - W\sin\alpha - \mu(W\cos\alpha - P\sin\beta) = 0,$$

$$\text{and } P\cos\beta - W\sin\alpha + \mu(W\cos\alpha - P\sin\beta) = 0,$$

hence the limits between which P must lie are

$$P = W\frac{\sin\alpha + \mu\cos\alpha}{\cos\beta + \mu\sin\beta}, \text{ and } P = W\frac{\sin\alpha - \mu\cos\alpha}{\cos\beta - \mu\sin\beta}.$$



3. Let MKL represent the isosceles triangle on the inclined plane; G its centre of gravity.

First suppose that the triangle begins to slide. Let R be the reaction of the plane perpendicular to itself, μ the coefficient of friction, and ϕ the inclination of the plane when sliding commences.

Resolving the forces parallel to the plane, we have

$$\mu R = W\sin\phi \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and resolving at right angles to the plane,

$$R = W\cos\phi \quad\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

from (1) and (2) we have

$$\mu = \tan\phi.$$

Next suppose that the triangle rolls over the corner K; then the vertical through G will pass through K when ϕ has received the proper value; draw GH at right angles to the plane; then

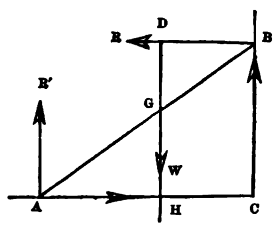
$$HK = \frac{1}{2}, \quad GH = \frac{\sqrt{3} \times 1\frac{1}{2}}{3} = \frac{\sqrt{3}}{2};$$

$$\therefore \tan\phi = \tan KGH = \frac{1}{2} \div \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}.$$

Hence if μ be less than $\frac{HG}{HK}$, sliding will take place before rolling. If $\mu > \frac{GH}{HK}$, rolling will take place before sliding; and for rolling and sliding to take place simultaneously, we must have $\mu = \frac{GH}{HK}$, or

$$\mu = \frac{1}{\sqrt{3}} = \tan 30^\circ.$$

4. Let AB be the rod, G its centre of gravity, which (since the rod is uniform) will be at its middle point; R the reaction of the plane CB; R' that of the plane CA.



Let the angle $BAC = \theta$. Then the forces acting on the rod are, its weight (W) acting at G in the vertical GH, R' at A, R at B, and the friction.

Resolving parallel to BC, we have

$$W - R' - \mu R = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

Resolving parallel to CA, we have

$$R - \mu' R' = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

Also, taking the moments round G, we have

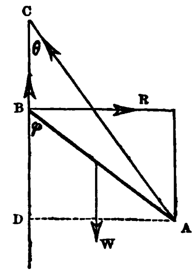
$$R\sin\theta + \mu R\cos\theta - R'\cos\theta + \mu' R'.\sin\theta = 0 \ldots\ldots (3).$$

From equations (1) and (2) we have

$$R = \frac{W\mu'}{1+\mu\mu'}, \quad R' = \frac{W}{1+\mu\mu'};$$

and substituting these values in equation (3), we get

$$\tan\theta = \frac{1 - \mu\mu'}{2\mu'}.$$



8. Let θ and ϕ represent the inclination to the vertical of the string and beam respectively; and let l = length of the beam, c the length of the string, and μ the coefficient of friction. Suppose the beam to be on the point of sliding downwards, then it is maintained in its position by the reaction (R) of the plane in the direction BR; by the friction μR acting in the direction BC; by the weight of the beam acting vertically downwards at its middle point, or centre of gravity (the beam being uniform); and by the tension of the string in the direction AC.

Resolving perpendicularly to the string, we have

$$R\cos\theta + \mu R\sin\theta = W\sin\theta.$$

Taking moments about c, if $CB = x$,

$$Rx = W \times \frac{1}{2} l \sin\phi;$$

$$\therefore \frac{\cos\theta + \mu\sin\theta}{x} = \frac{2}{l} \cdot \frac{\sin\theta}{\sin\phi} = \frac{2}{c};$$

$$\therefore \cos\theta + \mu\sin\theta = 2\frac{x}{c} = 2\frac{\sin(\phi - \theta)}{\sin\phi} = 2(\cos\theta - \cot\phi\sin\theta);$$

$$\therefore \mu + 2\cot\phi = \cot\theta.$$

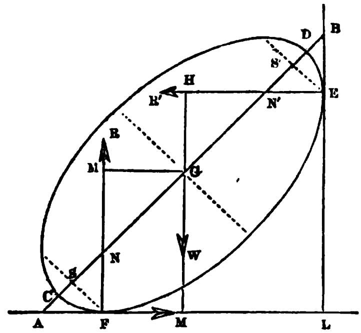
Now $\sin\phi = \frac{c}{l}\sin\theta$; $\therefore \cot^2\phi = \operatorname{cosec}^2\phi - 1 = \frac{l^2}{c^2}\operatorname{cosec}^2\theta - 1$;

$$\therefore c^2\cot^2\phi = l^2\cot^2\theta + l^2 - c^2;$$

$$\therefore c^2(\cot^2\theta - 2\mu\cot\theta + \mu^2) = 4c^2\cot^2\phi = 4l^2\cot^2\theta + 4l^2 - 4c^2;$$

$$\therefore (4l^2 - 4c^2 - \mu^2c^2)\tan^2\theta - 2\mu c^2\tan\theta + 4l^2 - c^2 = 0.$$

If the beam be on the point of sliding downwards, we must in this formula write $-\mu$ for μ.



9. Let $CFED$ be a section of the cylinder through its centre of gravity G; CD its major axis inclined at an angle of $45°$ to the horizon and equal to $2a$.

The forces acting upon the cylinder are the reaction (R') of the plane BL acting in the direction EH; the reaction (R) of the plane AL acting in the direction FK; the friction (μR) of the horizontal plane acting at F in the direction FM; and the weight of the cylinder acting at its centre of gravity G in the vertical GM.

Resolving the forces horizontally, we have

$$R' - \mu R = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (1).$$

Taking the moments round G, we get

$$R' \times GH - R \times GK + \mu R \times GM = 0 \quad \dots\dots\dots \quad (2).$$

The tangents at E and F intersect at right angles in L, a point in the minor axis produced. Let $GS = x$, $SF = y$;

$$\text{then since } \angle A = 45°, \quad \frac{b^2 x}{a^2 y} = \tan 45° = 1; \quad \therefore \frac{y}{x} = \frac{b^2}{a^2}.$$

$$\text{Now } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \quad \therefore \frac{x^2}{a^2}\left(1 + \frac{a^2 y^2}{b^2 x^2}\right) \text{ or } \frac{x^2}{a^2}\left(1 + \frac{b^2}{a^2}\right) = 1;$$

$$\therefore x = \frac{a^2}{(a^2 + b^2)^{\frac{1}{2}}},$$

$$GK = GN \sin 45° = e^2 x \sin 45° = \frac{a^2 e^2}{\sqrt{2}\,(a^2 + b^2)^{\frac{1}{2}}} = GH.$$

$$\text{Also } GM = MA = \frac{1}{2} AL = \frac{1}{2}(AG^2 + GL^2)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left\{\left(\frac{a^2}{x}\right)^2 + \left(\frac{b^2}{y}\right)^2\right\}^{\frac{1}{2}} = \frac{1}{2}\left\{\frac{a^4}{x^2}\left(1 + \frac{b^4 x^2}{a^4 y^2}\right)\right\}^{\frac{1}{2}} = \frac{1}{2}\frac{a^2}{x}\sqrt{2} = \frac{(a^2 + b^2)^{\frac{1}{2}}}{\sqrt{2}}.$$

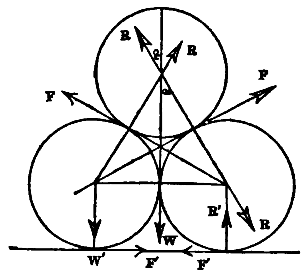
Now since by (1) $R' = \mu R$; therefore by substitution &c. in (2), we have

$$(\mu - 1)\frac{a^2 e^2}{\sqrt{2}\,(a^2+b^2)^{\frac{1}{2}}} + \mu\frac{(a^2+b^2)^{\frac{1}{2}}}{\sqrt{2}} = 0;$$

$$\therefore\ \mu\,(a^2e^2 + a^2 + b^2) = a^2e^2;$$

$$\therefore\ \mu = \frac{e^2}{e^2+1+1-e^2} = \frac{1}{2}e^2.$$

11. The forces as indicated in the diagram.



Let μ, μ' be the coefficients of friction respectively between the cylinders, and each cylinder and the plane.

For the upper cylinder we have

$$W - 2R\cos\alpha - 2F\sin\alpha = 0 \ldots\ldots (1).$$

The other two equations of this cylinder are identical. For one of the lower cylinders

$$W' - R' + R\cos\alpha + F\sin\alpha = 0 \ldots\ldots\ldots\ldots\ldots (2),$$

$$F' - R\sin\alpha + F\cos\alpha = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

$$F' - F = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

From (3) and (4) we have

$$\frac{F}{R} = \frac{\sin\alpha}{1+\cos\alpha} = \tan\frac{\alpha}{2},$$

and from (1) and (2) we get

$$2R' = 2W' + W.$$

Also, from (1), (3), and (4), we get

$$F' = \frac{W\sin\alpha}{2\,(1+\cos\alpha)} = \frac{W}{2}\tan\frac{\alpha}{2};$$

$$\therefore\ \frac{F'}{R'} = \frac{W\tan\frac{\alpha}{2}}{2W' + W}.$$

Now in order that the points of contact may all slip together we must have

$$\tan\frac{\alpha}{2}=\mu, \text{ and } \frac{W\tan\frac{\alpha}{2}}{2W'+W}=\mu';$$

$$\therefore \frac{W'}{W}=\frac{1}{2}\left(\frac{\mu}{\mu'}-1\right).$$

DYNAMICS.

The Collision or Impact of Bodies.

Ex. 1.

1. Let $v=$ common velocity after impact, then

$$v=\frac{Aa+Bb}{A+B}=\frac{96+35}{19}=\frac{131}{19}=6\tfrac{17}{19} \text{ ft. per second,}$$

velocity lost by $A=a-v=8-6\frac{17}{19}=1\frac{2}{19}$ ft. per second,

velocity gained by $B=v-b=6\frac{17}{19}-5=1\frac{17}{19}$ ft. per second.

3. Here $A=8$, $B=5$, $b=9$; $\therefore v=3\times 9$.

$$\text{Now } v=\frac{Aa+Bb}{A+b};$$

$$\therefore 27\times 13=8a+5\times 9;$$

$$\therefore a=\frac{27\times 13-5\times 9}{8}=\frac{9\times 34}{8}=38\tfrac{1}{4}\text{ ft.}$$

7. Given $A=\dfrac{4B}{3}$, $a=\dfrac{5b}{4}$, $\epsilon=1$;

$$\therefore\ u = \frac{Aa + Bb - Ba + Bb}{A+B} = \frac{(A-B)\,a + 2Bb}{A+B} = \frac{\frac{B}{3}a + \frac{B}{5}.\,8a}{\frac{7B}{3}}$$

$$= \frac{a\left(1 + \frac{24}{5}\right)}{7} = \frac{29a}{35} \dots\dots\dots (1),$$

$$\text{and } v = \frac{(B-A)\,b + 2Aa}{A+B} = \frac{\frac{8B}{3}a - \frac{B}{3}.\,\frac{4a}{5}}{\frac{7B}{3}} = \frac{8a - \frac{4a}{5}}{7} = \frac{36a}{35} \dots (2);$$

$\therefore$ from (1) and (2) we have

$$u : v = 29 : 36.$$

10. Let $x =$ the number of bodies.

The velocity communicated to the second body

$$= \frac{2Aa}{A + 3A} = \frac{a}{2},$$

and the velocity communicated to the third body

$$= \frac{2 \times 3A}{3A + 3^2A} \times \frac{a}{2} = \frac{a}{2^2}.$$

Similarly, we find the velocity of the last body $= \dfrac{a}{2^{x-1}}$;

$$\therefore\ \frac{a}{2^{x-1}} = \frac{a}{64}, \text{ by the hypothesis;}$$

$$\therefore\ 2^{x-1} = 64 = 2^6;$$

$$\therefore\ x - 1 = 6;$$

$$\therefore\ x = 7, \text{ the No. of balls.}$$

11. Let the weight of the intermediate ball be x lb.,

$$A = 2 \text{ lb.},\quad B = 8 \text{ lb.},\quad a = 9 \text{ ft.},\quad \epsilon = 1,$$

velocity communicated from A to $x = \dfrac{2Aa}{A+x}$,

$$\text{.................................}x \text{ to } B = \frac{2x\left(\dfrac{2Aa}{A+x}\right)}{x+B}$$

$$= \frac{4Aax}{(A+x)(x+B)};$$

$$\therefore\ 2 = \frac{4Aax}{(A+x)(x+B)};$$

$$\therefore\ (x+2)(x+8) = 2 \times 2 \times 9x,$$

$$x^2 - 26x + 13^2 = 169 - 16;$$

$$\therefore\ x = 13 \pm \sqrt{153} = 25{\cdot}369 \text{ or } {\cdot}63.$$

16. Velocity of A after impact $= a - \dfrac{(1+\epsilon)B(a+b)}{A+B}$.

$$\text{Given}\ \left.\begin{aligned} A &= \frac{5B}{6} \\ a &= 7 \\ b &= -5\tfrac{3}{5} \\ \epsilon &= \frac{2}{3} \end{aligned}\right\} \begin{aligned} u &= 7 - \frac{\left(1+\frac{2}{3}\right)B\,12.\frac{3}{5}}{\dfrac{11B}{6}} = 7 - \frac{10 \times 12\frac{3}{5}}{11} = 7 - \frac{126}{11}, \\ &= \frac{77-126}{11} = -\frac{49}{11} = -4{\cdot}4545\ \&\text{c.} \end{aligned}$$

velocity of B after impact $= \dfrac{(1+\epsilon)A(a+b)}{A+B} - b$;

$$\therefore\ v = \frac{\left(1+\frac{2}{3}\right)\frac{5B}{6}\,.\,12\frac{3}{5}}{\dfrac{11B}{6}} - 5\tfrac{3}{5} = \frac{\frac{5}{3} \times \frac{5}{6} \times \frac{63}{5}}{\dfrac{11}{6}} - 5\tfrac{3}{5}$$

$$= \frac{\frac{5}{3} \times 63 - \frac{308}{5}}{11} = \frac{1575 - 924}{11 \times 15} = \frac{651}{11 \times 15}$$

$$= \frac{217}{55} = 3{\cdot}9\dot{4}\dot{5}.$$

18. $$v = \frac{Aa - Bb + \epsilon A(a+b)}{A+B},$$

$$u = \frac{Aa - Bb - \epsilon B(a+b)}{A+B};$$

therefore the relative velocity

$$= v - u = \frac{\epsilon(A+B)(a+b)}{A+B} = \epsilon(a+b).$$

Hence the required distance = relative velocity × time

$$= \frac{2}{3}(25+16) \times \frac{9}{2} = 3 \times 41 = 123 \text{ ft.}$$

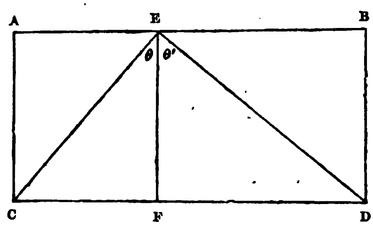
20. Let θ, θ' be the angles of incidence and reflexion respectively.

$$\text{Then } \tan\theta' = \frac{1}{\epsilon}\tan\theta, \text{ and } \theta + \theta' = 90^\circ;$$

$$\therefore \tan(90^\circ - \theta) = 3\tan\theta;$$

$$\therefore \tan\theta = \frac{1}{\sqrt{3}} = \tan 30^\circ; \quad \therefore \theta = 30^\circ.$$

23. Let t, t' be the times of describing CE, ED with the velocities v, v' respectively.



$$\text{Then } \frac{t}{t'} = \frac{CE}{v} \div \frac{ED}{v'} = \frac{CE}{ED} \times \frac{v'}{v}$$

$$= \frac{\sin(90^\circ - \theta')}{\sin(90^\circ - \theta)} \times \frac{\sin\theta}{\sin\theta'} = \frac{\tan\theta}{\tan\theta'} = \frac{\epsilon}{1}.$$

24. Let θ, θ' be the angles of incidence and reflexion respectively.

$$\text{Then } \tan\theta' = \frac{1}{\epsilon}\tan\theta = \sqrt{3}\tan\theta, \text{ since } \epsilon = \tan 30°,$$

$$\sin\theta' = \frac{v}{v'}\sin\theta = \sqrt{2}\sin\theta, \qquad v' = v\sin 45°;$$

$$\therefore \cos\theta' = \sqrt{\frac{2}{3}}\cos\theta;$$

$$\therefore 1 = 2\sin^2\theta + \frac{2}{3}\cos^2\theta = \frac{4}{3}\sin^2\theta + \frac{2}{3};$$

$$\left.\begin{aligned} \therefore \sin\theta &= \frac{1}{2} = \sin 30° \\ \sin\theta' &= \frac{\sqrt{2}}{2} = \sin 45° \end{aligned}\right\}; \quad \theta = 30°, \ \theta' = 45°.$$

26. Let x be the magnitude of the intermediate body; then A impinging on x at rest communicates to it a velocity $= \frac{(1+\epsilon)Aa}{A+x}$; x impinging on B at rest communicates to it a velocity $= \frac{(1+\epsilon)x}{x+B} \cdot \frac{(1+\epsilon)Aa}{A+x}$; but A impinging on B at rest communicates a velocity $= \frac{(1+\epsilon)Aa}{A+B}$;

$$\therefore \frac{(1+\epsilon)^2 Aax}{(x+A)(x+B)} = \frac{(1+\epsilon)Aa}{A+B};$$

$$\text{whence } x^2 + (A+B)x + AB = (A+B)x + \epsilon(A+B)x;$$

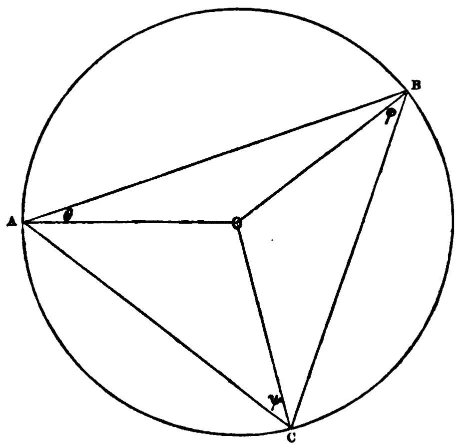
$$\therefore x^2 - \epsilon(A+B)x = -AB;$$

$$\text{whence } x = \frac{A+B}{2}\left\{\epsilon \pm \left(\epsilon^2 - \frac{4AB}{(A+B)^2}\right)^{\frac{1}{2}}\right\};$$

and for the expression under the vinculum to be possible, ϵ must be $> \frac{2(AB)^{\frac{1}{2}}}{A+B}$;

$$\therefore \text{the limits of } \epsilon \text{ are } 1 \text{ and } \frac{2(AB)^{\frac{1}{2}}}{A+B}.$$

28. Let A be the point from which the ball is projected; B and C the points of first and second reflexion; $\angle OAB=\theta$, $OBC=\phi$, $OCA=\psi$.



Then $2\theta+2\phi+2\psi=180°$; $\therefore\ \theta+\phi+\psi=90°$.

Now $\tan\phi=\frac{1}{\epsilon}\tan\theta$, and $\tan\psi=\frac{1}{\epsilon}\tan\phi=\frac{1}{\epsilon^2}\tan\theta$.

But $\tan\theta=\tan\{90°-(\phi+\psi)\}=\cot(\phi+\psi)=\dfrac{1}{\tan(\phi+\psi)}$

$$=\frac{1-\tan\phi\tan\psi}{\tan\phi+\tan\psi}=\frac{1-\dfrac{1}{\epsilon^3}\tan^2\theta}{\dfrac{1}{\epsilon}\tan\theta+\dfrac{1}{\epsilon^2}\tan\theta};$$

$$\therefore\ \tan^2\theta\left(\frac{1}{\epsilon}+\frac{1}{\epsilon^2}+\frac{1}{\epsilon^3}\right)=1;\quad \therefore\ \tan\theta=\left(\frac{\epsilon^3}{1+\epsilon+\epsilon^2}\right)^{\frac{1}{2}}.$$

Again, let t_1, t_3 be the times of describing AB, CA; and v_1, v_2, v_3 the velocities of the ball along AB, BC, CA respectively.

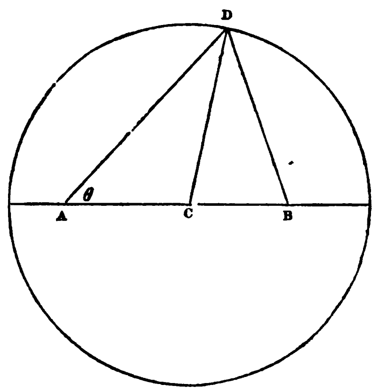
$$\text{Then } \frac{v_1}{v_2}=\frac{\sin\phi}{\sin\theta},\quad \frac{v_2}{v_3}=\frac{\sin\psi}{\sin\phi};\quad \therefore\ \frac{v_1}{v_3}=\frac{\sin\psi}{\sin\theta};$$

$$\text{and } \frac{t_1}{t_3}=\frac{AB}{v_1}\div\frac{CA}{v_3}=\frac{AB}{CA}\cdot\frac{v_3}{v_1}=\frac{\sin(\phi+\psi)}{\sin(\theta+\phi)}\times\frac{\sin\theta}{\sin\psi}=\frac{\cos\theta\sin\theta}{\cos\psi\sin\psi}$$

$$=\frac{\tan\theta\cos^2\theta}{\tan\psi\cos^2\psi}=\epsilon^2\times\frac{1+\tan^2\psi}{1+\tan^2\theta}=\epsilon^2\frac{1+\frac{1}{\epsilon^4}\left(\frac{\epsilon^3}{1+\epsilon+\epsilon^2}\right)}{1+\frac{\epsilon^3}{1+\epsilon+\epsilon^2}}$$

$$=\epsilon^2\frac{\epsilon+\epsilon^2+\epsilon^3+1}{\epsilon+\epsilon^2+\epsilon^3+\epsilon^4}=\epsilon.$$

29. Let $AC=a$, $CB=b$; $CD=r$ the radius, $\angle CAD=\theta$. Since the elasticity of the body is perfect, $\angle CDB=CDA$.



$$\text{Hence } \frac{AD}{DB}=\frac{AC}{CB}=\frac{a}{b},$$

$$\text{and } AD\times DB=AC\times CB+CD^2=ab+r^2;$$

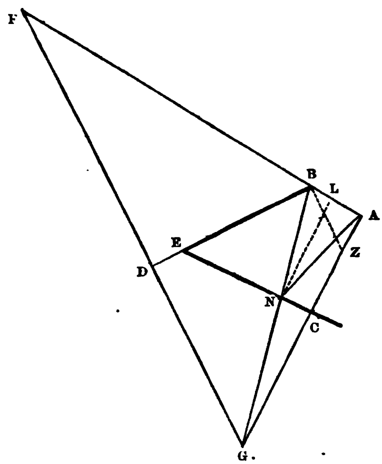
$$\therefore AD^2=\frac{a}{b}(ab+r^2).$$

$$\text{Now } \cos\theta=\frac{AD^2+AC^2-CD^2}{2AD\times AC}=\frac{\frac{a}{b}(ab+r^2)+a^2-r^2}{2a\left\{\frac{a}{b}(ab+r^2)\right\}^{\frac{1}{2}}}$$

$$=\frac{2a^2b+(a-b)r^2}{2a\{ab(ab+r^2)\}^{\frac{1}{2}}}.$$

30. 1°. Let the body be imperfectly elastic, and ϵ its elasticity.

Let BE, EC be the given planes, E the given point between them.



Draw AC perpendicular to the plane EC, and produce it to G, so that

$$AC : CG = \epsilon : 1.$$

Draw GD perpendicular to the plane BE, and produce it to F, making

$$GD : DF = \epsilon : 1.$$

Join AF cutting the plane BE in B, and BG cutting the plane EC in N. AF will be the direction of projection required, and the path described by the body will be $ABNA$.

Draw BZ perpendicular to the plane BE, and NL perpendicular to the plane EC; then

$$\tan ABZ : \tan NBZ = \tan GFA : \tan BGF = DG : DF = \epsilon : 1,$$

hence, if the body impinge in the direction AB, it will be reflected in the direction BN.

Again, $\tan BNL : \tan ANL = \tan BGA : \tan NAG = AC : CG = \epsilon : 1$, which shews that if the body impinge in the direction BN, it will be reflected in the direction NA, and will strike the body A.

2°. If the body be perfectly elastic,

$$AC = CG, \quad GD = DF.$$

31. In the expressions for u and v, make $b = 0$, $\epsilon = 1$, and we have

$$A\text{'s vel. after impact} = \frac{(A-B)\,a}{A+B}; \quad B\text{'s vel.} = \frac{2Aa}{A+B};$$

therefore by question,

$$\left(\frac{A-B}{A+B}\right) Aa = \frac{Aa}{4}; \quad \therefore 4(A-B) = A+B;$$

$$\therefore A : B = 5 : 3 \dots\dots\dots\dots\dots\dots (1).$$

Now, if in the expression $\left(\frac{A-B}{A+B}\right)\times a$ we put A for B, B for C, and $\frac{2Aa}{A+B}$ for a, we shall have

$$B\text{'s vel. after striking } C = \frac{B-C}{B+C}\times\frac{2Aa}{A+B};$$

$$\therefore\ B\text{'s momentum} = \frac{B(B-C)}{(A+B)(B+C)}\times 2Aa;$$

$$\therefore\ \frac{B(B-C)}{(A+B)(B+C)}\times 2Aa = \frac{Aa}{4};\quad \therefore\ \frac{8B}{A+B}\times\frac{B-C}{B+C} = 1 \ldots (2),$$

$$\text{from (1) and (2), } \frac{8B}{\frac{1}{3}.\,8B} = \frac{B+C}{B-C},\ \text{whence } 2B = 4C,$$

$$\text{or } B : C = 2 : 1 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Similarly we find the velocity of C after striking D

$$= \frac{C-D}{C+D}\times\frac{2BC}{B+C}\times\frac{2Aa}{A+B} = \frac{Aa}{4},$$

which, combined with equations (1) and (3), gives

$$\frac{C-D}{C+D} = \frac{1}{2},\ \text{ or } C : D = 3 : 1 \ldots\ldots\ldots\ldots (4),$$

therefore from (1), (3), and (4) we have

$$A : B : C : D = 10 : 6 : 3 : 1.$$

If there be n balls, we have, as before,

$$\frac{A-B}{A+B}\times Aa = \frac{Aa}{n};\quad \therefore\ \frac{A-B}{A+B} = \frac{1}{n};\quad \therefore\ \frac{A}{B} = \frac{n+1}{n-1},$$

$$\frac{B-C}{B+C}\times B\left(\frac{2Aa}{A+B}\right) = \frac{Aa}{n};\quad \therefore\ \frac{B-C}{B+C} = \frac{A+B}{2nB} = \frac{1}{n-1};$$

$$\therefore\ \frac{B}{C} = \frac{n}{n-2};\ \text{ similarly } \frac{C}{D} = \frac{n-1}{n-3}.$$

Generally, if m_r, m_{r+1} be the masses of the r^{th} and $(r+1)^{\text{th}}$ balls, we obtain $\frac{m_r}{m_{r+1}} = \frac{n-r+2}{n-r}$.

33. Let $m_1 = m_2 = m_3 = \ldots = m_n$ be the masses of the n spheres, a, v_1, $v_2 \ldots v_{n-1}$ their velocities, respectively.

$$\text{Then } v_1 = \frac{m_1 a}{m_1 + m_2} = \frac{a}{2},$$

$$v_2 = \frac{(m_1 + m_2)\, v_1}{m_1 + m_2 + m_3} = \frac{2}{3} \times \frac{a}{2} = \frac{a}{3},$$

$$v_3 = \frac{(m_1 + m_2 + m_3)\, v_3}{m_1 + m_2 + m_3 + m_4} = \frac{3}{4} \times \frac{a}{3} = \frac{a}{4}$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\text{hence } v_{n-2} = \frac{a}{n-1}.$$

Now, the time which elapses

before the 2d sphere is put in motion $= \dfrac{l}{a}$,

.... 3d .... $= \dfrac{l}{a} + \dfrac{2l}{a}$,

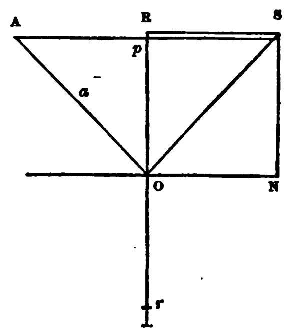
.... 4th .... $= \dfrac{l}{a} + \dfrac{2l}{a} + \dfrac{3l}{a}$,

.... $= \ldots\ldots\ldots\ldots$

.... nth .... $= \dfrac{l}{a} + \dfrac{2l}{a} + \dfrac{3l}{a} + \ldots + \dfrac{(n-1)\, l}{a}$

$$= \frac{n(n-1)}{1 \cdot 2} \cdot \frac{l}{a}.$$

36. Let AO be the direction in which the body A is moving; BO, that in which B is moving; $\angle AOB = \alpha$. Let AO, OB represent the velocities of A and B respectively; $\therefore AO = BO$. Resolve velocity AO into two others, Ap, pO, respectively perpendicular and parallel to B's direction; then the resolved velocity Ap is not affected by the collision of the bodies.



Take therefore

$$ON\,(= Ap) = a \sin \alpha,$$

which will represent A's velocity after impact in direction Ap.

The bodies therefore impinge *directly* on each other, A with vel. $= -a\cos\alpha$; B with vel. $= -a$;

therefore A's vel. in direction OR

$$=\frac{-a\cos\alpha-a+\epsilon a\cos\alpha-\epsilon a}{2}=-\frac{a}{2}\{1+\epsilon+(1-\epsilon)\cos\alpha\}=OR;$$

therefore A's vel. after impact

$$=\{OR^2+ON^2\}^{\frac{1}{2}}=\left[\frac{a^2}{4}\{1+\epsilon+(1-\epsilon)\cos\alpha\}^2+a^2\sin^2\alpha\right]^{\frac{1}{2}};$$

$$v=B\text{'s vel. after impact}=\frac{-a\cos\alpha-a-\epsilon a\cos\alpha+\epsilon a}{2};$$

$$\therefore\ v^2=\frac{a^2}{4}\{1-\epsilon+(1+\epsilon)\cos\alpha\}^2.$$

The expression for A's (vel.)^2 may be put under the form

$$a^2\left\{\cos^2\frac{\alpha}{2}+\epsilon\sin^2\frac{\alpha}{2}\right\}^2+4a^2\left(\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}\right)^2,$$

which is to be a maximum; therefore

$$2\left\{\cos^2\frac{\alpha}{2}+\epsilon\sin^2\frac{\alpha}{2}\right\}\cdot\left\{2\epsilon\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}-2\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}\right\}$$
$$-3\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}\left(\cos\frac{2\alpha}{2}-\sin\frac{2\alpha}{2}\right)=0,$$

which, reduced, gives

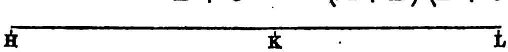
$$\tan^2\frac{\alpha}{2}\{2+\epsilon-\epsilon^2\}=1+\epsilon,\quad\text{or}\ \tan^2\frac{\alpha}{2}=\frac{1}{2-\epsilon};$$

$$\therefore\ \alpha=2\cot^{-1}(2-\epsilon)^{\frac{1}{2}}.$$

41. Let a be the velocity of A before impact,
u .... after impinging on B,
v .... B after impact of A,
w .... impinging on C.

Then $u=\dfrac{A-B}{A+B}\times a,\quad v=\dfrac{2Aa}{A+B},$

$$w=\frac{B-C}{B+C}\times v=\frac{2A(B-C)a}{(A+B)(B+C)}.$$



Let H, K be the positions of B and C when at rest, L the point where A overtakes B; $HK=c$, $KL=x$.

$$\text{Then time of } A\text{'s moving over } HL=\frac{c+x}{u},$$

$$\text{.........} B\text{'s ....} =\frac{c}{v}+\frac{x}{w}.$$

$$\text{Hence } \frac{c}{u}+\frac{x}{u}=\frac{c}{v}+\frac{x}{w};\quad \therefore\ x\left(1-\frac{u}{w}\right)=c\left(\frac{u}{v}-1\right);$$

$$\therefore\ x=c\left(\frac{A-B}{2A}-1\right)\div\left\{1-\frac{(A-B)(B+C)}{2A(B-C)}\right\}$$

$$=c\frac{(A+B)(B-C)}{(A-B)(B+C)-2A(B-C)}=c\frac{(A+B)(B-C)}{3AC-B(A+B+C)};$$

$$\therefore\ HL=x+c=c\frac{2C(A-B)}{3AC-B(A+B+C)}.$$

43. Let A, B, be the bodies; a, b, their respective velocities before impact, and v their common velocity after impact, then

$$Aa+Bb=(A+B)v;\quad \therefore\ A(a-v)=B(v-b)\ \text{......}\ (1).$$

Let $l=$ vel. lost by $A=a-v$; $g=$ vel. gained by $B=v-b$,

$$\text{then } a-b=l+g;\quad \therefore\ (a+v)-l=(b+v)+g\ \text{......}\ (2).$$

From equations (1) and (2), we have

$$A(a^2-v^2)-Al(a-v)=B(v^2-b^2)+Bg(v-b),$$

$$\text{or } Aa^2+Bb^2-(A+B)v^2=Al(a-v)+Bg(v-b)$$

$$=Al^2+Bg^2,$$

which proves the proposition.

44. $u=m$'s velocity after impact,

and $v=m'$'s velocity after impact,

$$\text{then } u=a-\frac{(1+\epsilon)m'(a-b)}{m+m'};\quad v=b+\frac{(1+\epsilon)m(a-b)}{m+m'}\text{......}(1);$$

$$\therefore\ v - u = b - a + \frac{(1+\epsilon)(m+m')(a-b)}{m+m'}$$

$$= (b-a) + (1+\epsilon)(a-b) \ldots\ldots\ldots (2).$$

Now by the Third Law of Motion

$$ma + m'b = mu + m'v,$$

and from (2) $\quad a + u = v + b + (1-\epsilon)(a-b)$;

$$\therefore\ ma^2 - mu^2 = m'v^2 - m'b^2 + (1-\epsilon)\, m'(a-b)(v-b),$$

or $m'v^2 + mu^2 = ma^2 + m'b^2 - (1-\epsilon)\, m'(a-b)(v-b) \ldots\ldots (3)$,

and from equations (3) and (1) we have

$$mu^2 + m'v^2 = ma^2 + m'b^2 - \frac{(1-\epsilon^2)\, mm'(a-b)^2}{m+m'}.$$

45. Let α, β be the angles which the directions of the velocities of the bodies, before impact, make with the line joining their centres at the moment of collision, then

$$u^2 = \frac{(A-B)^2 a^2 \cos^2\alpha - 4B(A-B)\, a\cos\alpha . b\cos\beta + 4B^2 b^2 \cos^2\beta}{(A+B)^2} + a^2 \sin^2\alpha,$$

$$v^2 = \frac{(A-B)^2 b^2 \cos^2\beta + 4A(A-B)\, a\cos\alpha . b\cos\beta + 4A^2 a^2 \cos^2\alpha}{(A+B)^2} + b^2 \sin^2\beta.$$

Multiplying (1) by A, (2) by B, and adding, we get

$$Au^2 + Bv^2 = \frac{(A+B)^2 \{Aa^2 \cos^2\alpha + Bb^2 \cos^2\beta\}}{(A+B)^2} + Aa^2 \sin^2\alpha + Bb^2 \sin^2\beta$$

$$= Aa^2 + Bb^2,$$

which proves the proposition.

46. Let $x = B$'s velocity after impact of A.

Then $u = \dfrac{Aa + Bb - B(a-b)}{A+B}$, since $\epsilon = 1$,

$$x = \frac{Aa + Bb + A(a-b)}{A+B};$$

$$\therefore\ x - u = \frac{(A+B)(a-b)}{A+B} = a - b; \quad \therefore\ a + u = x + b.$$

Also $Aa + Bb = Au + Bx$; $\therefore A(a-u) = B(x-b)$,

$A(a^2-u^2) = B(x^2-b^2)$; $\therefore Aa^2 + Bb^2 = Au^2 + Bx^2$.

In like manner we may prove that $Bx^2 + Cc^2 = Bv^2 + Cw^2$;

$$\therefore Aa^2 + Bb^2 + Cc^2 = Au^2 + Bv^2 + Cw^2.$$

Uniformly Accelerated Motion and Gravity.

Ex. 2.

1. Here $\left.\begin{array}{l} t = 11 \text{ sec.} \\ g = 32{\cdot}2 \text{ ft.}\end{array}\right\}$; $\therefore s = \frac{1}{2}gt^2 = 16{\cdot}1 \times 121 = 1948{\cdot}1$ ft.

$$v = gt = 32{\cdot}2 \times 11 = 354{\cdot}2 \text{ ft.}$$

3. Given $s = 450$ ft., $g = 32{\cdot}2$ ft.;

$$\therefore v = (2s \times g)^{\frac{1}{2}} = (900 \times 32{\cdot}2)^{\frac{1}{2}} = (28980)^{\frac{1}{2}} = 170{\cdot}2 \text{ ft.}$$

Momentum $= 170{\cdot}2 \times 140 = 23828$ lbs. $=$ 10 tons $12\frac{3}{4}$ cwt.

5. Here $s = 1320$ ft.; $\therefore t = \left(\frac{2s}{f}\right)^{\frac{1}{2}} = \left(\frac{1320}{g}\right)^{\frac{1}{2}}$,

or $t = 9{\cdot}05$ sec., the time for which the body has been falling;

$\therefore$ space described in the last second $= \frac{1}{2}(2t-1)$

$$= \frac{1}{2}(18{\cdot}1 - 1) = 275{\cdot}4 \text{ ft.}$$

8. Let $t =$ whole time of descent;

$\therefore$ the altitude of the tower $= \frac{1}{2}gt^2$,

and the space described in $(t-1)$ seconds $= \frac{1}{2}g(t-1)^2$;

$\therefore$ the space described in the last second $= \frac{1}{2}g(2t-1)$,

and, by the question, $\frac{1}{2}gt^2 = 3\,\frac{1}{2}g(2t-1)$;

$$\therefore\ t^2=6t-3;$$

$$\therefore\ t=3+\sqrt{6}=5\cdot449 \text{ sec.};$$

$$\therefore\ \text{the altitude of the tower}=\frac{1}{2}gt^2=477\cdot9 \text{ ft.}$$

10. Let g, g' be the measures of the force of gravity at the two places; t the time of a body describing a feet, and v the velocity acquired at the place where g is the force of gravity.

$$\text{Then } t=\left(\frac{2a}{g}\right)^{\frac{1}{2}}, \quad \text{and} \quad v=(2ag)^{\frac{1}{2}},$$

$$t-T=\left(\frac{2a}{g'}\right)^{\frac{1}{2}}, \quad v+m=(2ag')^{\frac{1}{2}};$$

$$\therefore\ T=(2a)^{\frac{1}{2}}\left(\frac{1}{\sqrt{g}}-\frac{1}{\sqrt{g'}}\right), \qquad m=(2a)^{\frac{1}{2}}(\sqrt{g'}-\sqrt{g}),$$

$$=\left(\frac{2a}{g}\right)^{\frac{1}{2}}\left(1-\sqrt{\frac{g}{g'}}\right), \qquad =(2ag')^{\frac{1}{2}}\left(1-\sqrt{\frac{g}{g'}}\right).$$

$$\text{Hence } mT=2a\sqrt{\frac{g'}{g}}\left(1-\sqrt{\frac{g}{g'}}\right)^2=2a\sqrt{\frac{g'}{g}}\left(1-2\sqrt{\frac{g}{g'}}+\frac{g}{g'}\right);$$

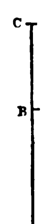
$$\therefore\ \frac{g}{g'}-2\left(1+\frac{mT}{4a}\right)\sqrt{\frac{g}{g'}}+\left(1+\frac{mT}{4a}\right)^2=-1+\left(1+\frac{mT}{4a}\right)^2$$

$$=\frac{mT}{2a}+\frac{m^2T^2}{16a^2};$$

$$\therefore\ \frac{g}{g'}=\left\{\left(1+\frac{mT}{4a}\right)\pm\frac{mT}{4a}\left(1+\frac{8a}{mT}\right)^{\frac{1}{2}}\right\}^2.$$

14. Let B be the given point; then

time up BC = time down CB;



$\therefore$ let $2t$ be the given time, and let $AB=a$,

$$\text{then } BC=CB=\frac{gt^2}{2};$$

$$\therefore\ AC=\frac{gt^2}{2}+a;$$

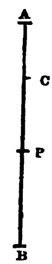
and the velocity of projection = vel. acquired in falling down CA;

$$\therefore\ V^2 = 2gs = 2g\left(\frac{gt^2}{2} + a\right);$$

$$\text{time down } CA = \frac{V}{g} = \left\{\frac{2a + gt^2}{g}\right\}^{\frac{1}{2}};$$

$$\therefore \text{ whole time of motion } (T) = 2\left\{\frac{2}{g}\left(a + \frac{1}{2}gt^2\right)\right\}^{\frac{1}{2}}.$$

15. Let A be the top of the spire, C the top of the steeple, B the bottom of the steeple, P the point where the bodies would meet.



Let $AB = a = 190$ ft., $CB = b = 150$ ft., $BP = x$,

then $AC = 40$ ft. $= a - b$, $AP = a - x$, $CP = b - x$.

Now the time of descending through $CP = \left(\frac{b-x}{\frac{1}{2}g}\right)^{\frac{1}{2}}$;

and the time of ascending through $BP =$ time down AB $-$ time down $AP = \left(\frac{g}{\frac{1}{2}g}\right)^{\frac{1}{2}} - \left(\frac{a-x}{\frac{1}{2}g}\right)^{\frac{1}{2}}$; but the time down CP must be equal to the time up BP; hence

$$\left(\frac{b-x}{\frac{1}{2}g}\right)^{\frac{1}{2}} = \left(\frac{a}{\frac{1}{2}g}\right)^{\frac{1}{2}} - \left(\frac{a-x}{\frac{1}{2}g}\right)^{\frac{1}{2}},$$

$$\text{or } (a-x)^{\frac{1}{2}} = a^{\frac{1}{2}} - (b-x)^{\frac{1}{2}},$$

$$\text{whence } 2(ab - ax)^{\frac{1}{2}} = b;\ \therefore\ ab - ax = \frac{b^2}{4},$$

$$\text{whence } x = b - \frac{b^2}{4a} = b\left(1 - \frac{b}{4a}\right)$$

$$= 150 \times \frac{61}{76} = \frac{9150}{76} = 120{\cdot}4.$$

16. Here $s = 125$, $V = 90$,

and from the formula $Vt + \frac{1}{2}ft^2 = s$,

we have, by substitution,

$$90t + 16{\cdot}1t^2 = 125,$$

whence $t = 1{\cdot}15$ sec. $=$ the time in which the body projected downwards will reach the middle point of the line.

Also from the formula $Vt - \frac{1}{2}ft^2 = s$, we find

$t' = 2{\cdot}57$ sec. = the time in which the body which is projected upwards will reach the middle point.

$\therefore\ t' - t = 1{\cdot}42$ sec., the time which must elapse before the body is projected downwards.

17. Let x = height of tower in feet,

then V (vel. of projection) $= \sqrt{2g\,(1{\cdot}8x)}$.

Hence $x = Vt - \frac{1}{2}gt^2 = 2\sqrt{3{\cdot}6gx} - 2g$, since $t = 2$ sec,

$$x - 2\sqrt{3{\cdot}6gx} + 3{\cdot}6g = (3{\cdot}6 - 2)\,g = 1{\cdot}6g;$$

$$\therefore\ x = \{\sqrt{3{\cdot}6} + \sqrt{1{\cdot}6}\}^2 g = \left(\frac{6+4}{\sqrt{10}}\right)^2 g = 10g = 321{\cdot}9 \text{ ft.}$$

18. Space described in p sec. $= Vp + \frac{1}{2}fp^2$,

space described in $(p-1)$ sec. $= V(p-1) + \frac{1}{2}f(p-1)^2$;

$\therefore$ space described in the p^{th} sec. $= V + \frac{1}{2}f(2p-1) = P$......(1),

and similarly,

space described in the q^{th} sec. $= V + \frac{1}{2}f(2q-1) = Q$.....(2);

$$\therefore\ f = \frac{P-Q}{p-q};$$

$$\therefore\ V = P - \frac{(P-Q)(2p-1)}{2(p-q)} = \frac{Q(2p-1) - P(2q-1)}{2(p-q)}.$$

20. Let x = the distance, from the upper extremity, at which they will meet,

then $at + \frac{1}{2}gt^2 = x$.......................(1),

and $ct - \frac{1}{2}gt^2 = h - x$ ....(2);

$$\therefore\ (a+c)\,t = h,\quad t = \frac{h}{a+c}.$$

Substituting in (1) we get

$$x = \frac{ah}{a+c} + \frac{g}{2} \cdot \frac{h^2}{(a+c)}$$
$$= \frac{h}{2} \cdot \frac{2a^2 + 2ac + gh}{(a+c)^2}.$$

22. Let u be the velocity of projection, $\epsilon = \frac{1}{2}$;
then,

vel. before striking the ceiling $= (u^2 - 2g \times 12)^{\frac{1}{2}}$;

$\therefore$ vel. after .... $= \epsilon\,(u^2 - 2g \times 12)^{\frac{1}{2}}$,

vel. before .... floor $= \{\epsilon^2\,(u^2 - 2g \times 12) + 2g \times 12\}^{\frac{1}{2}}$,

vel. after .... $= \epsilon\,\{\epsilon^2(u^2 - 2g \times 12) + 2g \times 12\}^{\frac{1}{2}}$.

Hence the vel. on again reaching the ceiling

$$= \{\epsilon^4\,(u^2 - 2g \times 12) + \epsilon^2\,(2g \times 12) - 2g \times 12\}^{\frac{1}{2}} = 0;$$
$$\therefore\ \epsilon^4 u^2 = (\epsilon^4 - \epsilon^2 + 1)\,2g \times 12;$$
$$\therefore\ u = (1 - 4 + 16)^{\frac{1}{2}}\sqrt{24g} = 2\sqrt{6 \times 13g} = 100{\cdot}216 \text{ ft.}$$

27. Let v_1, v_2, v_3...... be the velocities of the ball just *before* striking the plane in the 1st, 2d, 3d...... descents;

s_1, s_2, s_3 ...... the heights through which the ball successively falls, ϵ the elasticity $= \frac{3}{5} = {\cdot}6$.

Then $v_1 = (2g \times 50)^{\frac{1}{2}}$; $s_1 = 50$,

$$v_2 = \epsilon v_1; \qquad \therefore\ s_2 = \frac{v_2^2}{2g} = \frac{\epsilon^2\,(2g \times 50)}{2g} = 50\epsilon^2,$$

$$v_3 = \epsilon v_2 = \epsilon^2 v_1; \ \therefore\ s_3 = \frac{v_3^2}{2g} = \frac{\epsilon^4\,(2g \times 50)}{2g} = 50\epsilon^4;$$

&c. &c. &c.

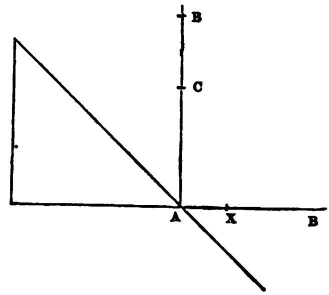
$\therefore$ whole space described

$= s_1 + 2s_2 + 2s_3 + \ldots\ldots$ to infinity

$= 100\,(1 + \epsilon^2 + \epsilon^4 + \ldots\ldots) - 50$

$$= 50\left(\frac{2}{1-\epsilon^2} - 1\right) = 50\left(\frac{1+\epsilon^2}{1-\epsilon^2}\right) = 50 \times \frac{1{\cdot}36}{{\cdot}64} = 50 \times 2\tfrac{1}{8} = 106\tfrac{1}{4} \text{ ft.}$$

29. Let D and C be the points from which the balls are dropped; $AD = a$, $AC = b$, and the distance AB on the horizontal plane over which the bodies have moved before collision takes place $= x$.



Since the velocity of each ball after impact equals its velocity before impact;

$$\therefore \left(\frac{2a}{g}\right)^{\frac{1}{2}} + \frac{x}{(2ag)^{\frac{1}{2}}} = \text{whole time} = \left(\frac{2b}{g}\right)^{\frac{1}{2}} + \frac{x}{(2bg)^{\frac{1}{2}}};$$

$$\therefore x\left\{\frac{1}{(2b)^{\frac{1}{2}}} - \frac{1}{(2a)^{\frac{1}{2}}}\right\} = (2a)^{\frac{1}{2}} - (2b)^{\frac{1}{2}};$$

$$\therefore x = (4ab)^{\frac{1}{2}} = 2\,(ab)^{\frac{1}{2}}.$$

30. Let O be the point where the bodies m, m' meet

$$PQ = 2a, \quad PO = x;$$

$$\therefore OQ = 2a - x,$$

h = height due to the velocity of m' on being reflected at Q.

Then velocity of m acquired down $PO = \sqrt{2gx}$,

.... m' on reaching $O = \sqrt{2g\,(h - 2a + x)}$.

Now the velocities of m, m' after impact must be equal to their velocities before impact, and have opposite signs;

$$\therefore -\sqrt{2gx}$$

$$= \frac{m\sqrt{2gx} - m'\sqrt{2g\,(h-2a+x)} - m'\left\{\sqrt{2gx} + \sqrt{2g\,(h-2a+x)}\right\}}{m + m'};$$

$$\therefore 2m\sqrt{2gx} = 2m'\sqrt{2g\,(h - 2a + x)};$$

$$\therefore x\,(m'^2 - m^2) = m'^2\,(2a - h);$$

$$\therefore PO \text{ or } x = \frac{m'^2}{m'^2 - m^2}\,(2a - h).$$

Obs. To satisfy the conditions of the problem, the momenta of m and m' *before* or *after* impact must be equal to one another; hence,

$$m\sqrt{2gx}=m'\sqrt{2g\,(h-2a+x)},$$

from which x is found as above.

Ex. 3.

1. Let $\theta=$ the inclination of the plane,

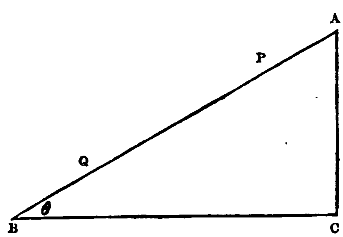
$$\text{then } s=\frac{1}{2}gt^2\sin\theta,$$

$$\text{or } 9=\frac{1}{2}g\sin\theta;$$

$$\therefore\ \sin\theta=\frac{18}{g}=\sin 33^\circ\ 59'\ 56''\cdot 4.$$

5. The accelerating force down $AC=g$,

.... $AB=g\sin\theta$.



$$\text{Hence time down } AC=\left(\frac{2AC}{g}\right)^{\frac{1}{2}},$$

$$\text{and } \ldots\ldots\ldots\ldots\ AB=\left(\frac{2AB}{g\sin\theta}\right)^{\frac{1}{2}};$$

$$\therefore\ \left(\frac{2AB}{g\sin\theta}\right)^{\frac{1}{2}}=3\left(\frac{2AC}{g}\right)^{\frac{1}{2}};\quad\therefore\ \frac{AC\sin\theta}{AB}=\frac{1}{9};$$

$$\therefore\ \sin^2\theta=\frac{1}{9};\quad\therefore\ \theta=\sin^{-1}\left(\frac{1}{3}\right)=19^\circ\ 28'\ 16''.$$

6. In the figure to (5) let $AP=x$, $PQ=AC=h$;

$$AB=l=40 \text{ ft.}$$

Then time down $AQ=\left\{\frac{2(x+h)}{g\sin\theta}\right\}^{\frac{1}{2}}$, where $\theta=30°$,

$$\text{............} \quad AP=\left(\frac{2x}{g\sin\theta}\right)^{\frac{1}{2}},$$

$$\text{............} \quad AC=\left(\frac{2h}{g}\right)^{\frac{1}{2}};$$

$$\therefore \left\{\frac{2(x+h)}{g\sin\theta}\right\}^{\frac{1}{2}}-\left(\frac{2x}{g\sin\theta}\right)^{\frac{1}{2}}=\left(\frac{2h}{g}\right)^{\frac{1}{2}};$$

$$\therefore (x+h)^{\frac{1}{2}}=x^{\frac{1}{2}}+(h\sin\theta)^{\frac{1}{2}}=x^{\frac{1}{2}}+\frac{h}{l^{\frac{1}{2}}};$$

$$\therefore x+h=x+\frac{h^2}{l}+2h\left(\frac{x}{l}\right)^{\frac{1}{2}};$$

$$\therefore x=\frac{(l-h)^2}{4l}=\frac{20^2}{4\times 40}=\frac{20}{8}=2\tfrac{1}{2} \text{ ft.}$$

9. Let h be the height and l the length of the inclined plane;

x be the space through which W has moved, when P ceases to act;

t be the time during which W has moved, when P ceases to act;

v be the velocity acquired by W when P ceases to act;

f be the accelerating force on the system.

$$\text{Then } f=\frac{\left(P-W\frac{h}{l}\right)g}{P+W}=\frac{(Pl-Wh)g}{(P+W)l}.$$

Since the velocity v is just sufficient to take W to the top of the plane, it is equal to the velocity acquired by a body moving through the space $l-x$ down the plane, the accelerating force being $f'=g\times\frac{h}{l}$.

$$\text{Hence } 2fx = v^2 = 2f'(l-x);$$

$$\therefore\ x = \frac{f'l}{f+f'} = \frac{P+W}{P}\left(\frac{hl}{h+l}\right);$$

$$t = \left(\frac{2x}{f}\right)^{\frac{1}{2}} = \frac{(P+W)\,l}{P}\left\{\frac{2Ph}{(Pl-Wh)(h+l)\,g}\right\}^{\frac{1}{2}}.$$

11. Let x = space described by the body projected from the bottom, then $l-x$ = space described by the body projected from the top, of the plane whose length is l, and inclination is α.

t = time of motion from the bottom,

then $n+t$ = ... top,

f = accelerating force along the plane $= g \sin\alpha$.

$$\text{Hence } x = ct - \frac{1}{2}ft^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

$$l - x = a(n+t) + \frac{1}{2}f(n+t)^2;$$

$$\therefore\ l = an + (a+c)\,t + \frac{1}{2}fn^2 + fnt,$$

whence $t = \dfrac{l - an - \frac{1}{2}fn^2}{a+c+fn}$, which substituted in (1) gives x.

13. Let l be the length of the plane, and θ its inclination, t the time of a body's falling down the plane.

$$\text{Then } t = \left(\frac{2l}{g\sin\theta}\right)^{\frac{1}{2}} = \left(\frac{2a}{g\sin\theta\cos\theta}\right)^{\frac{1}{2}} = \left(\frac{4a}{g\sin 2\theta}\right)^{\frac{1}{2}}.$$

Hence t is least, when $\sin 2\theta$ is greatest;

$$\therefore\ \sin 2\theta = 1 = \sin 90^\circ;$$

$$\therefore\ \theta = 45^\circ, \text{ and therefore the height} = a.$$

15. Let h, k, l be the height, base, and hypothenuse respectively.

Then, the time down $(h) = \left(\frac{2h}{g}\right)^{\frac{1}{2}}$, and the vel. acquired $= (2gh)^{\frac{1}{2}}$;

$$\therefore \text{.........} \text{ along } (k) = \frac{k}{(2gh)^{\frac{1}{2}}},$$

$$\text{.........} \text{ down } (l) = \left(\frac{2l}{g\frac{h}{l}}\right)^{\frac{1}{2}} = \left(\frac{2l^2}{gh}\right)^{\frac{1}{2}};$$

$$\therefore \frac{l}{\sqrt{h}} = \sqrt{h} + \frac{k}{2\sqrt{h}}; \quad \therefore l = h + \frac{1}{2}k;$$

$$\therefore h^2 + hk + \frac{1}{4}k^2 = l^2 = h^2 + k^2;$$

$$\therefore \frac{h}{k} = \frac{3}{4}.$$

17. Let x = side of the second triangle,

f the accelerating force $= g \sin 60°$, the triangles being equilateral.

The velocity acquired down $(a) = \sqrt{2fa}$,

.... along the horizontal plane $= \sqrt{2fa} \times \cos 60°$,

.... after impact at the base of the second triangle $= \sqrt{2fa} \times \cos^2 60°$,

.... which being just sufficient to take the body to the vertex of the second Δ = velocity acquired down its side $(x) = \sqrt{2fx}$;

$$\therefore x = a \cos^4 60° = \frac{a}{16}.$$

Again, the time down $(a) = \left(\frac{2a}{f}\right)^{\frac{1}{2}}$,

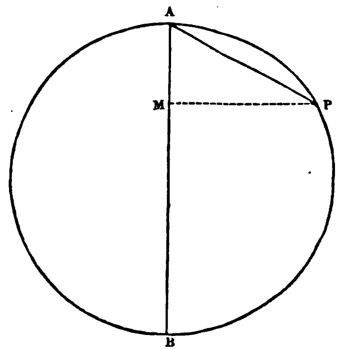
$$\text{..........} \text{ along } (d) = \frac{d}{\sqrt{2fa} \times \cos 60°},$$

$$\text{..........} \text{ up } (x) = \left(\frac{2x}{f}\right)^{\frac{1}{2}} = \left(\frac{a}{8f}\right)^{\frac{1}{2}};$$

therefore whole time

$$=\left\{(2a)^{\frac{1}{2}}+\frac{(2a)^{\frac{1}{2}}}{4}+\frac{2d}{(2a)^{\frac{1}{2}}}\right\}\frac{1}{\sqrt{f}}=\left(\frac{5}{2}a+2d\right)\div(ga\sqrt{3})^{\frac{1}{2}}.$$

19. Let ABP be a circle, AB a vertical diameter, PA any chord drawn through A, then velocity acquired in falling through AB is



$$(2g\times AB)^{\frac{1}{2}}\ldots\ldots\ldots\ldots(1),$$

vel. acquired down AP

$$=(2g\times AM)^{\frac{1}{2}},$$

and $AP^2=AM.AB$;

$$\therefore\ AM=\frac{AP^2}{AB};$$

therefore vel. acquired down AP

$$=\left(\frac{2g.AP^2}{AB}\right)^{\frac{1}{2}}$$

$$=AP\left(\frac{2g}{AB}\right)^{\frac{1}{2}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

$$\text{therefore by the question } AP\times\left(\frac{2g}{AB}\right)^{\frac{1}{2}}=\frac{(2g\times AB)^{\frac{1}{2}}}{2};$$

$$\therefore\ AP=\frac{1}{2}AB,$$

$$\cos BAP=\frac{AP}{AB}=\frac{AP}{2AP}=\frac{1}{2}=\cos 60^\circ;$$

$$\therefore\ \angle BAP=60^\circ.$$

21. Let O be the centre of the circle, CA and CB the chords, such that $\angle COA=\theta$, $COB=2\theta$, then

$$\text{time down } AC=\left\{\frac{2AC}{g\sin OCA}\right\}^{\frac{1}{2}}=\left\{\frac{2AC}{g\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}\right\}^{\frac{1}{2}}=\left\{\frac{2AC}{g\cos\frac{1}{2}\theta}\right\}^{\frac{1}{2}},$$

$$\text{time down } BC = \left\{\frac{2BC}{g \sin OCB}\right\}^{\frac{1}{2}} = \left\{\frac{4AC\cos\frac{1}{2}\theta}{g\sin\left(\frac{\pi}{2}-\theta\right)}\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{4AC\cos\frac{1}{2}\theta}{g\cos\theta}\right\}^{\frac{1}{2}};$$

$$\text{and by the question } \left\{\frac{4AC\cos\frac{1}{2}\theta}{g\cos\theta}\right\}^{\frac{1}{2}} = n\left\{\frac{2AC}{g\cos\frac{1}{2}\theta}\right\}^{\frac{1}{2}};$$

$$\therefore \frac{4AC\cos\frac{1}{2}\theta}{g\cos\theta} = n^2 \cdot \frac{2AC}{g\cos\frac{1}{2}\theta};$$

$$\therefore 2\cos^2\frac{1}{2}\theta = n^2\cos\theta;$$

$$\therefore \cos\theta = n^2\cos\theta - 1;$$

$$\therefore \cos\theta\,(n^2-1) = 1;$$

$$\text{whence } \sec\theta = n^2 - 1.$$

23. The axis of the parabola, and the tangent at its vertex being taken as axes of x and y, and (h, k) being the co-ordinates of P, we have, l being latus rectum,

$$k^2 = lh.$$

Let the vertical through P meet the horizontal line at a distance l below A and M, then

$$\text{time down } PM = \left\{\frac{2(h+l)}{g}\right\}^{\frac{1}{2}},$$

$$\text{time down } PA = \left\{\frac{2PA^2}{gh}\right\}^{\frac{1}{2}} = \left\{\frac{2(h^2+k^2)}{gh}\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{2(h+l)}{g}\right\}^{\frac{1}{2}} = (\text{time down } PM).$$

25. Let θ be the inclination of the required diameters to the axis of x, and n seconds the given time, then the length of the diameters $= \dfrac{2b}{(e^2\cos^2\theta - 1)^{\frac{1}{2}}}$;

$$\therefore\ n = \left\{\frac{\dfrac{4b}{(e^2\cos^2\theta - 1)^{\frac{1}{2}}}}{g\sin\theta}\right\}^{\frac{1}{2}}; \quad \therefore\ n^4 = \frac{16b^2}{g^2(1-\cos^2\theta)(e^2\cos^2\theta - 1)};$$

$$\therefore\ n^4g^2(e^2+1)\cos^2\theta - n^4g^2e^2\cos^4\theta - n^4g^2 = 16b^2;$$

$$\therefore\ \cos^4\theta - \frac{e^2+1}{e^2}\cos^2\theta = -\frac{g^2n^4+16b^2}{g^2n^4e^2},$$

which quadratic solved gives for the position of the diameters

$$\cos\theta = \pm\left[\frac{e^2+1}{2e^2} \pm \left\{\left(\frac{e^2+1}{2e^2}\right)^2 - \frac{g^2n^4+16b^2}{g^2n^4e^2}\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}},$$

and when the time is a minimum,

$$\left\{\frac{4b}{g\sin\theta\,(e^2\cos^2\theta - 1)^{\frac{1}{2}}}\right\}^{\frac{1}{2}} \text{ is a minimum;}$$

$$\therefore\ \sin\theta\,(e^2\cos^2\theta - 1)^{\frac{1}{2}} \text{ is a maximum;}$$

$$\therefore\ e^2\cos\theta\,.\,\frac{\sin\theta}{(e^2\cos^2\theta - 1)^{\frac{1}{2}}} - \cos\theta\,.\,\frac{(e^2\cos^2\theta - 1)^{\frac{1}{2}}}{\sin\theta} = 0;$$

$$\therefore\ e^2\sin^2\theta - e^2\cos^2\theta + 1 = 0, \text{ or } 2e^2\cos^2\theta = e^2+1;$$

$$\therefore\ \cos\theta = \pm\left(\frac{e^2+1}{2e^2}\right)^{\frac{1}{2}}.$$

26. Let AB be the axis of the cycloid, AL the vertical through A meeting the horizontal plane in L, then

$$AL = AB\sin 60^\circ = a\frac{\sqrt{3}}{2}.$$

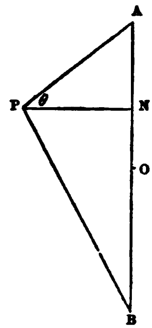
Let AP be either of the chords drawn from the vertex to one extremity of the base, then, by the property of the cycloid,

$$AP = (AB^2 + PB^2)^{\frac{1}{2}} = \left(a^2 + \frac{\pi^2a^2}{4}\right)^{\frac{1}{2}} = \frac{a}{2}(\pi^2+4)^{\frac{1}{2}};$$

$$\text{therefore time down } AP = \left\{\frac{2AP}{g\times AL}\right\}^{\frac{1}{2}} = \left\{\frac{a(\pi^2+4)}{g\sqrt{3}}\right\}^{\frac{1}{2}}.$$

27. Let $AB = 2a$, be bisected in O,

$ON = x$, $NP = y$.



Then time down $AP = \left(\frac{2AP}{g \sin\theta}\right)^{\frac{1}{2}} = \left(\frac{2AP^2}{gAN}\right)^{\frac{1}{2}}$

$$= \left(\frac{2}{g}\right)^{\frac{1}{2}} \left\{\frac{y^2 + (a-x)^2}{a-x}\right\}^{\frac{1}{2}}$$

.... $PB = \left(\frac{2PB^2}{gNB}\right)^{\frac{1}{2}}$

$$= \left(\frac{2}{g}\right)^{\frac{1}{2}} \left\{\frac{y^2 + (a+x)^2}{a+x}\right\}^{\frac{1}{2}}.$$

If c be the constant, then

$$\frac{y^2 + (a-x)^2}{a-x} + \frac{y^2 + (a+x)^2}{a+x} = \frac{g}{2} \times c;$$

$$\therefore\ y^2 \times \frac{2a}{a^2 - x^2} + 2a = \frac{cg}{2};$$

$$\therefore\ y^2 = \left(\frac{cg}{4a} - 1\right)(a^2 - x^2),$$

the equation, of which the locus is an ellipse or hyperbola, according as c is greater or less than $\frac{4a}{g}$.

29. The accelerating force f during the fall of 12 feet

$$= \frac{2g}{2+9} = \frac{2}{11}g.$$

V_1, vel. acquired $= (2fs)^{\frac{1}{2}} = \left\{2\left(\frac{2}{11}g\right) \times 12\right\}^{\frac{1}{2}} = 11{\cdot}852$ ft.

t_1, the time $= \left(\frac{2s}{f}\right)^{\frac{1}{2}} = \left(\frac{24 \times 11}{2g}\right)^{\frac{1}{2}} = 2{\cdot}025$ sec.

Let v_1, v_2 be the velocities acquired when the 2 lb. has fallen through 15 ft. and 20 ft. respectively;

t the time of describing the first 15 feet,

t' .... last 5 feet.

Then $v_1 = \left\{2\left(\frac{2g}{11}\right) \times 15\right\}^{\frac{1}{2}} = 13\cdot25$ ft.

$$v_2^2 = v_1^2 + 2g \times 5 = 2g\left(\frac{30}{11} + 5\right) = \frac{170}{11}g;$$

$$\therefore\ v_2 = 22\cdot304 \text{ ft.},$$

$$t = \left(\frac{2 \times 15}{\frac{2}{11}g}\right)^{\frac{1}{2}} = \left(\frac{165}{g}\right)^{\frac{1}{2}} = 2\cdot264 \text{ sec.}$$

Now $v_2 = v_1 + gt'$; $\therefore\ t' = \dfrac{v_2 - v_1}{g} = \cdot281$ sec.

$$\therefore\ t_2 = t + t' = 2\cdot545 \text{ sec.}$$

31. Let $9 + x$ and $9 - x$ lb. be the weights of the extremities of the chord.

$$\text{Then } f = \frac{(9+x) - (9-x)}{(9+x) + (9-x)}g = \frac{x}{9}g;$$

$$\therefore\ 3 \times 13 = \frac{1}{2}\left(\frac{x}{9}g\right) \times 13^2, \text{ since } s = \frac{1}{2}ft.;$$

$$\therefore\ x = \frac{6 \times 9}{13g} = \cdot12904.$$

Hence the weights are $9\cdot12904$ and $8\cdot87096$ lb.

33. Let f be the accelerating force on the bodies P and Q,

and f' .... $P-p$ and Q.

Then $f = \dfrac{P-Q}{P+Q}g$; and $f' = \dfrac{Q-P+p}{Q+P-p}g$;

V, the velocity acquired by P in descending a feet,

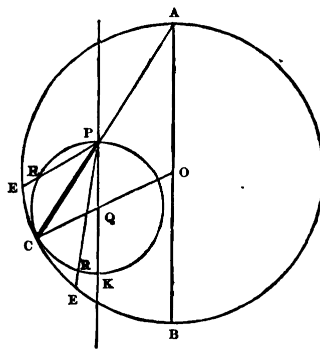
$$= \left(\frac{P-Q}{P+Q} \times 2ga\right)^{\frac{1}{2}};$$

therefore the space through which $P-p$ will descend $= \dfrac{V^2}{2f'}$

$$= \frac{(P-Q)(Q+P-p)}{(P+Q)(Q-P+p)}a;$$

after which $P-p$ will ascend under the action of f'.

35. Let O be the centre of the given circle ACB, AB a vertical diameter, P the given point. Draw PK parallel to AB; join PA, and produce AP to meet the circle in C; join OC cutting PK in Q; PC is the straight line required.



The triangles CPQ, CAO, will be similar, and since $CO = OA$, $\therefore$ $CQ = QP$; therefore a circle PCK described with centre Q, and radius QC, will pass through P, and since OQ joins the centres of the circles it will pass through the point of contact; therefore, as the circle PCK touches the circle ACB in the point C, it can touch it in no other point.

Draw PRE, any other line, from P to the circumference ABC;

then the time down $PE >$ time down PR,

.... $>$ .... PC,

hence PC is the line of quickest descent.

37. Let s be the focus, SP the radius vector, and θ its inclination to the axis major, then by the polar equation to the ellipse $SP = \dfrac{a(1-e^2)}{1 - e\cos\theta}$;

therefore time down $SP = \left\{\dfrac{2a(1-e^2)}{g\cos\theta\,(1 - e\cos\theta)}\right\}^{\frac{1}{2}} =$ a minimum;

$$\therefore \cos\theta\,(1 - e\cos\theta) \text{ is a maximum.}$$

Differentiating, $-\sin\theta\,(1 - e\cos\theta) + e\sin\theta\,.\cos\theta = 0$;

$$\therefore 2e\cos\theta - 1 = 0;$$

$$\therefore \cos\theta = \frac{1}{2e}, \text{ or } \theta = \cos^{-1}\left(\frac{1}{2e}\right).$$

MOTION UPON A CURVE, AND THE SIMPLE PENDULUM.

Ex. 4.

1. Let a, b, c be the lengths of the planes; then velocity acquired through $a = \sqrt{2ga}$, the resolved portion of which in the direction of B is $\sqrt{\frac{3}{2}ga}$;

$$\therefore \text{ vel. acquired through } b = \left\{\frac{3}{2}ga + \sqrt{3}\,gb\right\}^{\frac{1}{2}},$$

the resolved portion of which in the direction of C is

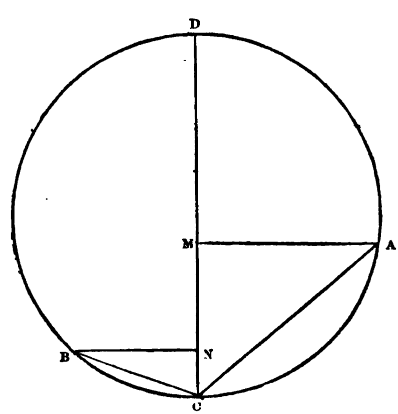
$$\sqrt{\frac{9}{8}ga + \frac{3\sqrt{3}}{4}gb};$$

$\therefore$ vel. through C

$$= \left(\frac{9}{8}ga + \frac{3\sqrt{3}}{4}gb + c\right)^{\frac{1}{2}} = \left\{\frac{g}{8}(9a + 6\sqrt{3}\,.\,b + 8c)\right\}^{\frac{1}{2}};$$

$$\therefore \text{ velocity required} = \frac{1}{4}\left\{\frac{3g}{2}(9a + 6\sqrt{3}\,b + 8c)\right\}^{\frac{1}{2}}.$$

2. Let the chords $AC = a$, $CB = b$, and $CD = d$;



$$\text{vel. acquired down } AC = \sqrt{2g \, . \, MC} = \sqrt{2g \times \frac{a^2}{d}},$$

$$\dots\dots\dots\dots\dots\dots\dots \quad BC = \sqrt{2g \, . \, NC} = \sqrt{2g \times \frac{b^2}{d}}.$$

The ball (m) which descends down the arc AC impinges directly on an equal ball at rest at C; hence, after impact, the velocity of the latter is equal to that which it would acquire in falling down the arc BC: now the

$$\text{velocity after impact} = \frac{m(1+e)\,a\sqrt{\frac{2g}{d}}}{m+m} = \frac{a}{2}(1+e)\sqrt{\frac{2g}{d}};$$

$$\therefore \frac{a}{2}(1+e)\sqrt{\frac{2g}{d}} = \sqrt{\frac{2g}{d} \times b^2};$$

$$\therefore e = \frac{2b}{a} - 1.$$

4. Let l be the length of the pendulum which vibrates 4 times in a second;

L be the length of the pendulum which vibrates once in a second.

$$\left.\begin{aligned} \text{Then } \frac{1}{4} &= \pi\sqrt{\frac{l}{g}} \\ \text{and } 1 &= \pi\sqrt{\frac{L}{g}} \end{aligned}\right\}; \qquad \begin{aligned} &\therefore \frac{1}{4} = \left(\frac{l}{L}\right)^{\frac{1}{2}}, \\ &\therefore l = \frac{1}{16} \times L = 2\text{·}4462 \text{ in.} \end{aligned}$$

5. Let g, g' measure the force of gravity at the earth's surface and at the distance of two of the earth's radii above its surface respectively; and t, t' the times of a vibration at these two stations.

$$\text{Then } t = \pi\sqrt{\frac{80}{g}}, \quad t' = \pi\sqrt{\frac{80}{g'}}; \quad \text{also } 1 = \pi\sqrt{\frac{L}{g}};$$

$$\therefore \frac{t'}{t} = \left(\frac{g}{g'}\right)^{\frac{1}{2}} = \frac{3r}{r}; \quad \text{also } \frac{t}{1} = \left(\frac{80}{L}\right)^{\frac{1}{2}};$$

$$\therefore t' = 3\left(\frac{80}{L}\right)^{\frac{1}{2}} = 4\text{·}289 \text{ sec.}$$

7. Let t = time in which the particle falls through 81 feet, then

$$t = \left(\frac{2 \times 81}{g}\right)^{\frac{1}{2}} = \left(\frac{162}{g}\right)^{\frac{1}{2}} \text{ seconds;}$$

$\therefore$ the pendulum makes 1 vibration in $\frac{1}{3}\left(\frac{162}{g}\right)^{\frac{1}{2}}$ sec.

Let l be the length of the pendulum, then

$$\pi\left(\frac{l}{g}\right)^{\frac{1}{2}} = \frac{1}{3}\left(\frac{162}{g}\right)^{\frac{1}{2}},$$

$$\text{whence } l = \frac{162}{9g} \times \frac{g}{\pi^2} = \frac{18}{\pi^2} = 1\cdot82378 \text{ feet.}$$

9. Let $l = L + 1\cdot05 = 40\cdot1893$ in.,

x = number of seconds lost in 12 hours.

$$\left.\begin{aligned} \text{Then } t &= \pi\sqrt{\frac{l}{g}} \\ 1 &= \pi\sqrt{\frac{L}{g}} \end{aligned}\right\}; \qquad \therefore t = \left(\frac{l}{L}\right)^{\frac{1}{2}}.$$

$$\text{Now } (43200 - x)\, t = 43200 \times 1 \text{ sec.}$$

$$\therefore 43200 - x = 43200\left(\frac{L}{l}\right)^{\frac{1}{2}} = 42631\cdot92;$$

$$\therefore x = 568\cdot08.$$

11. Let x be the required height in miles;

t be the time of a vibration at the earth's surface;

g, g' be the force of gravity at the lower and upper stations respectively.

$$\left.\begin{aligned} \text{Then } t &= \pi\sqrt{\frac{l}{g}} \\ 1 &= \pi\sqrt{\frac{l}{g'}} \end{aligned}\right\}; \quad \therefore t = \left(\frac{g'}{g}\right)^{\frac{1}{2}} = \frac{4000}{4000 + x}.$$

But $(60 \times 60 + 3)\, t = 60 \times 60 \times 1$ sec.;

$$\therefore\ 1 + \frac{x}{4000} = \frac{1}{t} = 1 + \frac{3}{3600};$$

$$\therefore\ x = \frac{4000}{1200} = 3\tfrac{1}{3} \text{ miles.}$$

13. Let x be the required height in miles;

t be the time of a vibration at the upper station;

g, g' be the force of gravity at the lower and upper stations respectively.

$$\text{Then}\ \left.\begin{aligned} 1 &= \pi\sqrt{\frac{L}{g}} \\ t &= \pi\sqrt{\frac{L}{g'}} \end{aligned}\right\};\qquad \therefore\ t = \left(\frac{g}{g'}\right)^{\frac{1}{2}} = \frac{4000 + x}{4000}.$$

Now $(60 \times 60 \times 24 - 48{\cdot}6)\, t = 60 \times 60 \times 24$ sec.;

$$\therefore\ 1 + \frac{x}{4000} = t = \frac{86400}{86351{\cdot}4} = 1 + \frac{48{\cdot}6}{86351{\cdot}4};$$

$$\therefore\ x = \frac{4000 \times 48{\cdot}6}{86351{\cdot}4} = 2\tfrac{1}{4} \text{ miles, nearly.}$$

15. Let g, g' be the force of gravity at the equator and pole respectively;

1, t be the times of a vibration at the equator and pole respectively.

$$\text{Then}\ \left.\begin{aligned} 1 &= \pi\sqrt{\frac{l}{g}} \\ t &= \pi\sqrt{\frac{l}{g'}} \end{aligned}\right\};\qquad \therefore\ t = \left(\frac{g}{g'}\right)^{\frac{1}{2}}.$$

Now $(86400 + 5 \times 60)\, t = 86400$ sec.; $\therefore\ t = \dfrac{288}{289}$;

$$\therefore\ g : g' = t^2 : 1 = 288^2 : 288^2 + 2 \times 288 + 1$$

$$= 144 : 144 + 1 + \frac{1}{2 \times 288}$$

$$= 144 : 145\tfrac{1}{576}.$$

18. Let g, g' be the force of gravity at Greenwich and the other place respectively.

Then $t=\pi\sqrt{\frac{L}{g'}}$ and $1=\pi\sqrt{\frac{L}{g}}$; $\therefore\ t=\left(\frac{g}{g'}\right)^{\frac{1}{2}}$.

$$\text{But } (86400-n)\,t = 86400 \times 1 \text{ sec.};$$

$$\therefore \frac{g}{g'}=t^2=\left(\frac{86400}{86400-n}\right)^2=\left(1-\frac{n}{86400}\right)^{-2}$$

$$=1+\frac{2n}{86400} \text{ nearly.}$$

19. At B, let $t=\pi\sqrt{\frac{l}{g}}$, and $\therefore\ 1=\pi\sqrt{\frac{l-\frac{1}{m}}{g}}$;

$$\therefore\ t=\left(\frac{l}{l-\frac{1}{m}}\right)^{\frac{1}{2}};\quad \therefore\ 1-\frac{1}{lm}=\frac{1}{t^2}.$$

$$\text{Now } (86400-60n)\,t=86400\times 1 \text{ sec.};$$

$$\therefore\ 1-\frac{1}{lm}=\left(1-\frac{n}{1440}\right)^2=1-\frac{n}{720} \text{ nearly};$$

$$\therefore\ l=\frac{720}{mn} \text{ nearly.}$$

22. Let x be the depth of the mine in miles;
g, g' be the force of gravity at the surface of the earth and in the mine.

Then $1=\pi\sqrt{\frac{L}{g}}$, and $t=\pi\sqrt{\frac{L}{g'}}$;

$$\therefore\ t=\left(\frac{g}{g'}\right)^{\frac{1}{2}}=\left(\frac{4000}{4000-x}\right)^{\frac{1}{2}},$$

since in the interior of the earth, gravity varies directly as the distance from the earth's centre.

$$\text{Now } (86400-10)\,t=86400\times 1 \text{ sec.},$$

$$1-\frac{x}{4000}=\frac{1}{t^2}=\left(1-\frac{1}{8640}\right)^2;$$

$$\therefore\ x=\frac{8000}{8640} \text{ nearly} = {\cdot}93 \text{ nearly.}$$

24. Let h = the height of the mountain, g' the force of gravity there, and $2m$ the number of seconds lost in a given time, then

$$g' = g\left\{\frac{r^2}{(r+h)^2}\right\}.$$

Also n, n' being the number of oscillations in the same time at the surface and up the mountain, we have

$$n : n' = \sqrt{g} : \sqrt{g'} = r+h : r;$$

$$\therefore \frac{n-n'}{n} = \frac{2m}{n} = \frac{h}{r+h} = \frac{h}{r} \text{ nearly};$$

$$\therefore h = \frac{2mr}{n}.$$

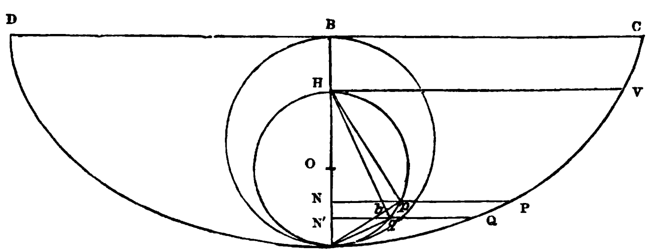
Let G be the force of gravity in the mine, d its depth, and n'' the number of oscillations there in the given time, then

$$G = g.\frac{r-d}{r} \text{ and } \frac{n}{n''} = \sqrt{\frac{g}{G}} = \sqrt{\frac{r}{r-d}};$$

$$\therefore \frac{n''}{n} = \frac{n-m}{n} = 1 - \frac{m}{n} = \sqrt{\frac{r-d}{r}} = 1 - \frac{d}{2r} \text{ nearly};$$

$$\therefore d = \frac{2rm}{n} = h.$$

26. Let V be the point from which the particle starts; VH horizontal meeting the axis in H. On AH describe a circle,



and let the ordinates PN, QN' of two contiguous points meet this circle in p, q; join Hp, Hq, and Ap cutting Hq in b.

$$\text{Now arc } AP = 2\sqrt{AB \,.\, AN}$$

$$= 2\sqrt{AB \,.\, \frac{Ap^2}{AH}}$$

$$= 2Ap\sqrt{\frac{AB}{AH}}.$$

$$\text{Similarly, arc } AQ = 2Aq\sqrt{\frac{AB}{AH}};$$

$$\therefore\ PQ = 2\,(Ap - Aq)\sqrt{\frac{AB}{AH}}.$$

Again, since the particle starts from rest at V, the velocity at P = vel. acquired in falling freely through the vertical height HN

$$= \sqrt{2g \,.\, HN} = \sqrt{2g \,.\, \frac{Hp^2}{AH}} = Hp \,.\, \sqrt{\frac{2g}{AH}};$$

and since PQ is very small, the velocity of the particle whilst describing PQ will be very nearly uniform and equal to its velocity at P, and the smaller PQ is taken, the more nearly will this supposition be true; also on the same supposition $Ap - Aq$ may be ultimately taken equal to bp.

$$\text{Hence time of describing } PQ = \frac{\text{arc } PQ}{\text{vel. at } P} \text{ ultimately,}$$

$$= 2\,(Ap - Aq)\left(\frac{AB}{AH}\right)^{\frac{1}{2}} \div Hp\left(\frac{2g}{AH}\right)^{\frac{1}{2}} = 2\,\frac{bp}{Hp}\left(\frac{AB}{2g}\right)^{\frac{1}{2}} = 2\,\angle pHq\left(\frac{AB}{2g}\right)^{\frac{1}{2}}.$$

Taking, therefore, the sum of successive small intervals starting from V, we get the time of describing VA, and the sum of the corresponding small angles is $= \angle\, VHA = \frac{\pi}{2}$; $\therefore$ if $AB = 2a$,

$$\text{the time down any arc } VA = \frac{\pi}{2}\left(\frac{4a}{g}\right)^{\frac{1}{2}} = \pi\left(\frac{a}{g}\right)^{\frac{1}{2}}.$$

Projectiles in a Non-resisting Medium.

Ex. 5.

1. $$\text{Range} = \frac{V^2}{g} \cdot \sin 2\alpha = \frac{60^2}{g} \sin 30^\circ$$

$$= \frac{1800}{32 \cdot 19084} = 55 \cdot 916 \text{ ft.}$$

$$H = \frac{V^2}{2g} \sin^2 \alpha = \frac{60^2}{2g} \sin^2 15^\circ = \frac{450}{g}(2 - \sqrt{3})$$

$$= \frac{120 \cdot 57714}{32 \cdot 19084} = 3 \cdot 7457 \text{ ft.}$$

$$T = \frac{2V}{g} \sin \alpha = \frac{120}{g} \sin 15^\circ$$

$$= \frac{31 \cdot 05828}{32 \cdot 19084} = \cdot 9648 \text{ sec.}$$

3. $$\left.\begin{aligned} R &= 2h \sin 2\alpha = 1000 \\ T &= \frac{2V}{g} \sin \alpha = 15 \text{ sec.} \end{aligned}\right\}, \text{ required } \alpha \text{ and } V.$$

Now $2V^2 \sin^2 \alpha = \dfrac{(15g)^2}{2}$,

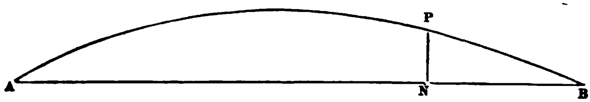
$$V^2 \sin 2\alpha = 1000g, \text{ since } V^2 = 2gh;$$

$$\therefore \tan \alpha = \frac{225}{2} g = \tan 74^\circ\ 33'\ 47'',$$

$$V = \frac{15g}{2 \sin \alpha} = 250 \cdot 46 \text{ ft.}$$

Also, $H = h \sin^2 \alpha = \dfrac{1000 \sin^2 \alpha}{2 \sin 2\alpha} = 250 \tan \alpha = 905 \cdot 35$ ft.

7. Let APB be the path of the projectile.



$$AN = \tfrac{1}{2}\text{ mile} = 2640\text{ ft.}$$

$$NP = 100\text{ ft.},\quad AB = 3600\text{ ft.}$$

Now since $y = x\tan\alpha - \dfrac{x^2}{4h\cos^2\alpha}$ is the equation to APB;

$$\left.\begin{aligned} \therefore\ 100 &= 2640\tan\alpha - \frac{2640^2}{4h\cos^2\alpha} \\ \text{and }\ 0 &= 3600\tan\alpha - \frac{3600^2}{4h\cos^2\alpha} \end{aligned}\right\},\ \text{required } \alpha \text{ and } V = \sqrt{2gh},$$

$$\therefore\ \frac{1}{4h\cos^2\alpha} = \frac{\tan\alpha}{3600}$$

$$\frac{100}{2640} = \tan\alpha\left(1 - \frac{2640}{3600}\right) = \frac{96}{360}\tan\alpha;$$

$$\therefore\ \tan\alpha = \frac{10}{264}\times\frac{30}{8} = \frac{25}{176} = \tan 8^\circ\, 5'\, 4'',$$

$$V^2 = g\times 2h = \frac{3600g}{2\tan\alpha\cos^2\alpha} = \frac{3600g}{\sin 2\alpha};$$

$$\therefore\ V = 645{\cdot}1\text{ ft.}$$

9. Since $\alpha = 45$, we have

$$T = \frac{V\sqrt{2}}{g},\qquad R = \frac{V^2}{g},$$

and the time in which the sound of the explosion was heard, after the ball had completed the range, is

$$\frac{V^2}{g}\div 35g = \frac{V^2}{35g^2};$$

$$\therefore\ \text{by question, } \frac{V^2}{35g^2} + \frac{V\sqrt{2}}{g} = 3\tfrac{1}{2},$$

and, completing the square of this quadratic,

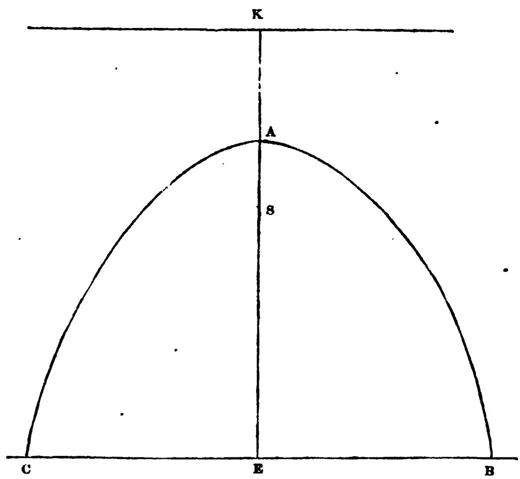
$$\left(\frac{V}{g}+\frac{35\times\sqrt{2}}{2}\right)^2=\frac{245}{2}+\frac{1225}{2}=\frac{2940}{4};$$

$$\therefore \frac{V}{g}+\frac{35\sqrt{2}}{2}=\frac{54{\cdot}221767}{2},$$

$$\text{whence } \frac{V}{g}=\frac{54{\cdot}221767-49{\cdot}497476}{2}=2{\cdot}362145;$$

$$\therefore R=\frac{V^2}{g}=(2{\cdot}362145)^2\times g=179{\cdot}616\text{ ft.}$$

11. Let S be the focus, and let the axis ES produced meet the directrix in K.



$$\text{Then } EK=h=\frac{V^2}{2g},$$

$$EA=H=h\sin^2\alpha;$$

$$\therefore AS=AK=EK-EA=h\cos^2\alpha.$$

$$\text{Hence, latus rectum}=4h\cos^2\alpha=\frac{2\,(850)^2}{g}\times\left(\frac{1}{2}\right)^2$$

$$=11222\text{ ft. nearly.}$$

$$\text{Also } CE \text{ or } x_1 = \frac{1}{2}R = h \sin 2\alpha = 9718\cdot 9 \text{ ft.},$$

$$\text{and } ES \text{ or } y_1 = h \sin^2\alpha - h\cos^2\alpha = h\left(\frac{3}{4} - \frac{1}{4}\right) = 5611\cdot 2 \text{ ft.}$$

14. Let $H = h\sin^2\alpha$, and $H' = h\sin^2 2\alpha$; then since the area of a parabola equals two-thirds of the circumscribing parallelogram if A, A' be the areas of the parabolas corresponding to α, 2α respectively,

$$A : A' = \frac{2}{3}RH : \frac{2}{3}RH' = \sin^2\alpha : \sin^2 2\alpha = 1 : 4\cos^2\alpha.$$

17. Since $\dfrac{H}{\frac{1}{2}R} = \dfrac{3}{4}$; $\therefore \dfrac{h\sin^2\alpha}{h\sin 2\alpha} = \dfrac{3}{4}$;

$$\therefore \tan\alpha = \frac{3}{2} = \tan 56^\circ\ 19'.$$

18. Let z = the height of the line; then

$$z = a\tan\alpha - \frac{a^2}{4h\cos^2\alpha} = a - \frac{a^2}{2h},$$

$$\text{and } o = (a+b) - \frac{(a+b)^2}{2h}, \quad \therefore \frac{1}{2h} = \frac{1}{a+b};$$

$$\therefore z = a - \frac{a^2}{a+b} = \frac{ab}{a+b}.$$

21. Let α' express the direction of projection of the second body;

t, t' be the times of describing the oblique range on the plane whose inclination to the horizon is β;

then

$$t = \frac{2V}{g}\cdot\frac{\sin(\alpha-\beta)}{\cos\beta}, \quad t' = \frac{2V}{g}\cdot\frac{\sin(\alpha'-\beta)}{\cos\beta};$$

$$\therefore \frac{t}{t'} = \frac{\sin(\alpha-\beta)}{\sin(\alpha'-\beta)}.$$

Now since the bodies strike the same point in the inclined plane,

$$4h\frac{\sin(\alpha-\beta)\cos\alpha}{\cos^2\beta}=4h\frac{\sin(\alpha'-\beta)\cos\alpha'}{\cos^2\beta};$$

$$\therefore\ 2\sin(\alpha-\beta)\cos\alpha=2\sin(\alpha'-\beta)\cos\alpha',$$

$$\sin(2\alpha-\beta)-\sin\beta=\sin(2\alpha'-\beta)-\sin\beta;$$

$$\therefore\ \sin(2\alpha-\beta)=\sin(2\alpha'-\beta)=\sin\{\pi-(2\alpha'-\beta)\};$$

$$\therefore\ 2\alpha-\beta=\pi-(2\alpha'-\beta),$$

$$\alpha'-\beta=\frac{\pi}{2}-\alpha;$$

$$\therefore\ \frac{t}{t'}=\frac{\sin(\alpha-\beta)}{\cos\alpha}.$$

25. Let the top of the tower be taken for the origin of co-ordinates,

$x=a=$ the distance of the part struck from the foot of the tower;

$-y=$ the height of the tower;

$$\text{then}\ -y=x\tan\alpha-\frac{1}{2}gt^2,$$

$$\text{or}\ y=\frac{1}{2}gn^2-a\tan 30^\circ$$

$$=\frac{n^2g}{2}-\frac{a}{\sqrt{3}}.$$

27. Take the top of the tower for the origin of co-ordinates, and let x, $-y$ be the co-ordinates of the point on the ground against which the body strikes, then

$$-y=x\tan\alpha-\frac{gx^2}{2V^2}(1+\tan^2\alpha);$$

$$\therefore\ \text{by the question,}\ -200=x\sqrt{3}-\frac{2gx^2}{50^2};$$

$$\text{whence}\ x^2-\frac{50^2\times\sqrt{3}}{2g}x=\frac{200\times 50^2}{2g},$$

which equation solved gives $x = \dfrac{16475\cdot635}{128\cdot76336} = 128$ feet;

also, if $t =$ the time of flight, we have

$$x = Vt\cos\alpha, \text{ or } 25t = 128;$$

$$\therefore\ t = \frac{128}{25} = 5\cdot12 \text{ sec.}$$

30. Given $\alpha = 75^\circ$, $i = 45^\circ$, $R' = 250$,

$$\text{and } R' = \frac{2V^2}{g}\cdot\frac{\sin(\alpha - i)\cos\alpha}{\cos^2 i};$$

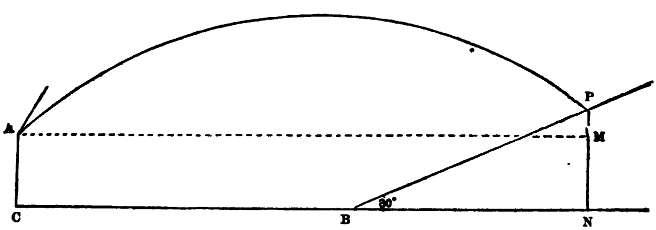
$$\therefore\ 250 = \frac{2V^2}{g}\cdot\frac{\sin 30^\circ.\cos 75^\circ}{\cos^2 45^\circ};$$

$$\therefore\ V = \left(\frac{125\times\cos^2 45^\circ}{g\sin 30^\circ.\cos 75^\circ}\right)^{\frac{1}{2}} = \left(\frac{500g}{\sqrt{6}-\sqrt{2}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{500g}{1\cdot035276}\right)^{\frac{1}{2}} = 124\cdot685,$$

$$T' = \frac{2V}{g}\cdot\frac{\sin 50^\circ}{\cos 45^\circ} = \frac{124\cdot685\times\sqrt{2}}{g} = 5\cdot47785 \text{ sec.}$$

31. Let the tower be denoted by $AC = 60$ ft., the width of the river by $CB = 300$ ft. The equation to the path of the projectile AP is



$$y = x\tan\alpha - \frac{x^2}{4h\cos^2\alpha} = x\sqrt{3} - \frac{x^2}{h}, \text{ since } \alpha = 60^\circ\ldots\ldots(1).$$

The equation to BP, the slope of the hill rising from the opposite bank A being the origin, is

$$y + 60 = (x - 300)\tan 30^\circ = (x - 300)\frac{1}{\sqrt{3}}\ldots\ldots\ldots(2);$$

To determine P, the intersection of AP and BP, whose co-ordinates are $AM = x_1$, $MP = y_1$, suppose; subtract (1) from (2),

$$\text{then } 60 = x_1\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right) + \frac{x_1^2}{h} - 100\sqrt{3}, \text{ where } h = 500 \text{ ft.};$$

$$\therefore\ x_1^2 - 2hx_1\frac{\sqrt{3}}{3} + \frac{h^2}{3} = (100\sqrt{3} + 60)\,h + \frac{h^2}{3} = 199935{\cdot}8;$$

$$\therefore\ x_1 = \frac{\sqrt{3}}{3}h \pm 447{\cdot}15 = 735{\cdot}8.$$

$$\text{Also } NP = (y_1 + 60) = (x_1 - 300)\frac{\sqrt{3}}{3}$$

$$= 435{\cdot}8 \times {\cdot}5773 = 251{\cdot}587.$$

Hence $BP = NP \operatorname{cosec} 30° = 2 \times 251{\cdot}587 = 503{\cdot}2$ ft. nearly.

33. Take the top of the plane AC for the origin of co-ordinates, and let x, $-y$, be the co-ordinates of D, and V the velocity with which the body leaves C; then

$$-y = x\tan\alpha - \frac{gx^2}{2V^2}(1 + \tan^2\alpha)\ldots\ldots\ldots\ldots(1),$$

$$\text{or } -\frac{AC}{\sqrt{2}} = x - \frac{g}{V^2}x^2, \text{ since } \alpha = 45°.$$

Let v be the velocity with which the body is projected up the plane AC, then by the question,

$$v^2 = 2g \times AC,$$

$$\text{but } V^2 = v^2 - 2g\sin 45° \times AC;$$

$$\therefore\ V^2 = 2g\,AC\left(1 - \frac{1}{\sqrt{2}}\right)\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Substituting in (1), we have

$$x^2 - 2AC\left(1 - \frac{1}{\sqrt{2}}\right)x = AC^2(\sqrt{2} - 1),$$

which equation solved gives

$$x = AC;$$

$$\therefore\ AD = AC + AC \times \frac{1}{\sqrt{2}} = 1{\cdot}7071 \times AC.$$

If t be the time up AC, then

$$V = v - g \sin 45^\circ . t;$$

$$\therefore\ t = \frac{v - V}{g \sin 45^\circ} = \left(\frac{AC}{g}\right)^{\frac{1}{2}} \{2 - (4 - 2\sqrt{2})^{\frac{1}{2}}\},$$

$$\text{and } T = \frac{x}{V \cos \alpha} = \frac{AC \times \sqrt{2}}{(2gAC)^{\frac{1}{2}} \left(1 - \frac{1}{\sqrt{2}}\right)^{\frac{1}{2}}}$$

$$= \left(\frac{AC}{g}\right)^{\frac{1}{2}} (2 + \sqrt{2})^{\frac{1}{2}},$$

the time of flight of the projectile;

therefore the whole time

$$= \left(\frac{AC}{g}\right)^{\frac{1}{2}} \{2 + (2 + \sqrt{2})^{\frac{1}{2}} - (4 - 2\sqrt{2})^{\frac{1}{2}}\}$$

$$= \left(\frac{AC}{g}\right)^{\frac{1}{2}} \times 2{\cdot}76537$$

$$= (AC)^{\frac{1}{2}} \times {\cdot}487 \text{ seconds.}$$

36. Take the top of the tower for the origin of co-ordinates, and let x, $-y$, be the co-ordinates of the point struck, (y being also the height of the tower), θ any angle of projection such that the body may strike the ground at the point in question, and V the velocity of projection; then

$$-y = x \tan \theta - \frac{x^2}{4h \cos^2 \theta} = x \tan \theta - \frac{x^2}{4h} (1 + \tan^2 \theta);$$

$$\text{whence } \tan^2 \theta - \frac{4h}{x} \tan \theta + 1 - \frac{4hy}{x^2} = 0 \ldots\ldots\ldots (1).$$

Let α, β be the directions in which the bodies are projected to strike the same point; then $\tan \alpha$ and $\tan \beta$ are the two roots of (1); therefore, by the Theory of Equations, we have

$$\tan \alpha + \tan \beta = \frac{4h}{x},$$

$$\text{and } \tan \alpha \tan \beta = 1 - \frac{4hy}{x^2} = 1 - \frac{y}{4h} (\tan \alpha + \tan \beta)^2;$$

$$\therefore\ y = 4h \cdot \frac{1 - \tan\alpha \tan\beta}{(\tan\alpha + \tan\beta)^2} = 4h \frac{\cos\alpha \cos\beta \cos(\alpha+\beta)}{\sin^2(\alpha+\beta)^2}$$

$$= 4h \cdot \frac{\cos\alpha \cos\beta \cot(\alpha+\beta)}{\sin(\alpha+\beta)}.$$

37. Take the point from which the bodies start for the origin of co-ordinates, the direction of A's motion being the axis of y, and let x, y be the co-ordinates of A, x_2, y_2 those of B at a given time t; then we have

$$x_1 = 0, \qquad y_1 = ut - \frac{1}{2}gt^2,$$

$$x_2 = vt\cos 30^\circ,\ y_2 = vt \sin 30^\circ - \frac{1}{2}gt^2.$$

Now by the nature of the centre of gravity, if $\bar{x}$, $\bar{y}$ be its co-ordinates at the time t, we have

$$\bar{x} = \frac{Ax_1 + Bx_2}{A+B} = \frac{\sqrt{3}\,Bvt}{2(A+B)} \text{ ....(1)},$$

$$\bar{y} = \frac{Ay_1 + By_2}{A+B} = \frac{A\left(ut - \frac{1}{2}gt^2\right) + B\left(\frac{1}{2}vt - \frac{1}{2}gt^2\right)}{A+B}$$

$$= \frac{Aut + \frac{1}{2}Bvt - (A+B)\cdot\frac{1}{2}gt^2}{A+B} \text{ ...(2)}.$$

From (1) we have

$$t = \frac{2(A+B)}{\sqrt{3}Bv}\bar{x},$$

which substituted in (2) gives for the equation to the path of the centre of gravity

$$\bar{y} = \frac{2Au + Bv}{\sqrt{3}Bv}\bar{x} - \frac{2(A+B)^2}{3B^2v^2}g\bar{x}^2.$$

39. Let α be the required angle of projection, then to find the value of α, we have (t being the time in which each body has moved before collision)

$$y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha} \text{ ....(1)},$$

$$t = \frac{x}{v\cos\alpha} \quad\text{.................................}(2),$$

$$y = ut - \frac{1}{2}gt^2 \quad\text{............................}(3);$$

from (1) and (3) we get

$$x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha} = ut - \frac{1}{2}gt^2 \quad\text{...........}(4),$$

and from (2) and (4) we have

$$tv\sin\alpha - \frac{1}{2}gt^2 = ut - \frac{1}{2}gt^2;$$

$$\text{whence } \sin\alpha = \frac{u}{v}.$$

40. Let x, y be the co-ordinates of any one of the bodies at the end of a given time t, then we have, θ being any angle of projection,

$$x = Vt\cos\theta \quad\text{..........................}(1),$$

$$y = Vt\sin\theta - \frac{1}{2}gt^2 \quad\text{...................}(2).$$

Now to find the locus of the bodies at the end of the given time t, we must eliminate the variable quantity θ from (1) and (2); to do this, add the squares of (1) and (2),

$$x^2 + \left(y + \frac{1}{2}gt^2\right)^2 = V^2t^2(\cos^2\theta + \sin^2\theta) = V^2t^2,$$

for the required locus, which is a circle.

42. Let β be the inclination of any one of the planes, as AP to the horizon, then by formula 5 we have

$$R' = 4h\,.\,\frac{\sin(\alpha-\beta)\cos\alpha}{\cos^2\beta},$$

and for this to be a maximum we must have

$$2\cos\alpha\sin(\alpha-\beta) \text{ a maximum,}$$

$$\text{or } \sin(2\alpha-\beta) - \sin\beta \text{ a maximum,}$$

and this will be the case when $\sin(2\alpha-\beta) = 1$,

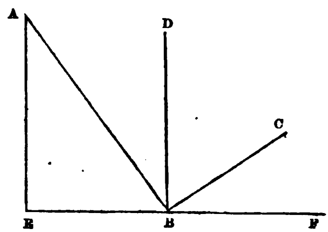
$$\text{or } 2\alpha - \beta = 90°,$$

and then we have

$$R' = \frac{2h}{1+\sin\beta} = \frac{2h}{1+\cos\theta},$$

if θ be the angle of projection with respect to the axis of y. Now this equation is the equation to a parabola whose parameter $=4h$, therefore the locus of P, P', P'', &c. is a parabola whose focus is the point of projection.

43. Let AB be the plane, l its length, α its inclination; and let $\theta = \angle CBF$, BC being the direction in which the body is reflected at B; E the elasticity.



Draw BD vertical. Then the velocity acquired in falling from A to $B = \sqrt{2gl\sin\alpha}$, and this being the velocity with which the body impinges against the plane BF at the angle of incidence $DBA = 90° - \alpha$, and the angle of reflexion being $90° - \theta$, by hypothesis, we have

$$\text{vel. of reflexion} = \frac{\sin(90°-\alpha)}{\sin(90°-\theta)} \cdot \sqrt{2gl\sin\alpha} = \frac{\cos\alpha}{\cos\theta} \cdot \sqrt{2gl\sin\alpha};$$

and this is the velocity with which the body is projected from B in the direction BC;

$$\therefore \text{ the range} = \frac{\cos^2\alpha}{\cos^2\theta} \cdot \frac{2gl\sin\alpha}{g} \cdot \sin 2\theta$$

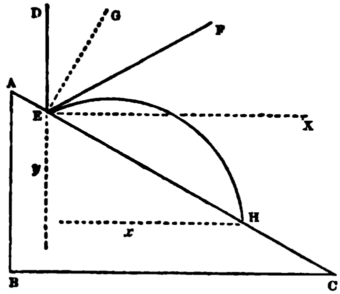
$$= 4l\cos^2\alpha\sin\alpha\tan\theta.$$

$$\text{But } \tan(90°-\theta) = \frac{1}{\epsilon}\tan(90°-\alpha), \text{ or } \tan\theta = \epsilon\tan\alpha;$$

$$\therefore \text{ the range} = 4l\epsilon\cos^2\alpha\sin\alpha\tan\alpha = \left(\frac{4}{3}\right)^{\frac{3}{2}}\epsilon l, \text{ since } \cos\alpha = \frac{1}{\sqrt{3}}.$$

45. Let AC be the plane, DE vertical, and suppose the body to fall down DE; draw GE perpendicular to the plane, and make the angle GEF equal to the angle DEG; EF will be the direction in which, since it is perfectly elastic, the ball will be reflected after impinging on the plane at E; and the velocity in direction $EF=$ vel. acquired in falling down

$$h = \sqrt{2gh}.$$



Suppose the ball after the rebound at E to strike the plane at H. Take E as the origin of co-ordinates, and let $x, -y$ be the co-ordinates of projection, then since

$$\text{angle of elevation} = FEX = 30^\circ, \quad V = \sqrt{2gh},$$

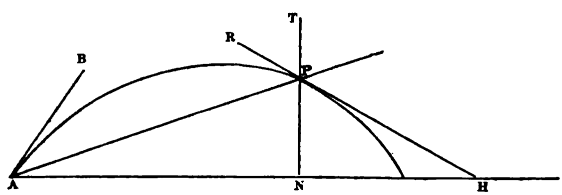
$$\text{and } y = x \tan 30^\circ = \frac{x}{\sqrt{3}},$$

$$\text{we have} \quad -\frac{x}{\sqrt{3}} = \frac{x}{\sqrt{3}} - \frac{gx^2}{4gh \times \frac{3}{4}}, \quad \text{whence } x = 2\sqrt{3}h;$$

$$\therefore\ EH = \sqrt{x^2 + y^2} = \sqrt{12h^2 + 4h^2} = 4h,$$

$$\text{and time} = \frac{x}{V \cos 30^\circ} = \frac{2\sqrt{3}h}{\sqrt{2gh} \times \frac{\sqrt{3}}{2}} = \left(\frac{8h}{g}\right)^{\frac{1}{2}}.$$

46. Let $XAB = \theta$, $XAP = \iota$.



Since the elasticity is perfect, and the body is reflected by the plane AP in the direction PT, which is vertical;

$$\therefore\ \angle RPT = 2\iota, \text{ and } \angle RHX = \frac{\pi}{2} + 2\iota.$$

$$\text{Now}\quad y = x\tan\theta - \frac{x^2}{4h\cos^2\theta}\ \dots\dots\dots\dots\dots(1);$$

$$\therefore\ \tan RHX = \frac{dy}{dx} = \tan\theta - \frac{x}{2h\cos^2\theta}.$$

$$\text{Also}\quad y = x\tan\iota\ \dots\dots\dots\dots\dots\dots\dots\dots(2).$$

Subtracting (2) from (1) we get

$$0 = (\tan\theta - \tan\iota)\,x - \frac{x^2}{4h\cos^2\theta};$$

$$\therefore\ 0 = 2(\tan\theta - \tan\iota) - \frac{x}{2h\cos^2\theta};$$

$$\therefore\ \tan RHX = \tan\theta - 2(\tan\theta - \tan\iota) = -\tan\theta + 2\tan\iota.$$

$$\text{But}\quad \tan RHX = \tan\left(\frac{\pi}{2} + 2\iota\right) = -\cot 2\iota;$$

$$\therefore\ \tan\theta = 2\tan\iota + \cot 2\iota = \frac{1}{2}(3\tan\iota + \cot\iota).$$

48. Let θ and ϕ be the angles which the tangents to the parabola at the points P and Q make with the horizon; t, t' the times from the point of projection to the same points; v the velocity of projection, and α the elevation.

At the end of the time t, the horizontal velocity of the body is $v\cos\alpha$, and its vertical velocity is $v\sin\alpha - g.t$. Wherefore if v' be the velocity of the body at the point P, we have

$$v'\cos\theta = \text{the horizontal vel. at } P = v\cos\alpha,$$

$$v'\sin\theta = \dots\ \text{vertical}\ \dots\dots\dots\dots = v\sin\alpha - gt;$$

$$\therefore\ \tan\theta = \frac{v\sin\alpha - gt}{v\cos\alpha}.$$

$$\text{Similarly}\quad \tan\phi = \frac{v\sin\alpha - gt'}{v\cos\alpha};$$

$$\therefore\ \tan\theta - \tan\phi = \frac{g}{v\cos\alpha}.(t' - t);$$

$$\text{and}\ \therefore\ t' - t \propto \tan\theta - \tan\phi.$$

50. When the ball has reached its greatest height, all its vertical velocity will have been destroyed; therefore, if V be the velocity of projection, α the elevation, and ϵ the elasticity of the ball, it will strike the plane with vel. $= V\cos\alpha$, and rebound in a horizontal direction with velocity $= \epsilon V \cos\alpha$.

The equation to the curve first described is

$$y = x\tan\alpha - \frac{x^2}{4h\cos^2\alpha},$$

whence the co-ordinates of the point of impact are obtained, viz.

$$y_1 = h\sin^2\alpha, \quad \text{and} \quad x_1 = h\sin 2\alpha.$$

If the origin of co-ordinates for the equation to the curve described after impact be at the point of impact, then in

$$y = x\tan\theta - \frac{x^2}{4h'\cos^2\theta}$$

we have $\theta = 0$, $h' = \dfrac{(\epsilon V\cos\alpha)^2}{2g} = h\epsilon^2\cos^2\alpha$, and $y = -y_1$;

$$\therefore\ -h\sin^2\alpha = -\frac{x^2}{4h\,\epsilon^2\cos^2\alpha}; \quad \therefore\ x_2 = h\epsilon\sin 2\alpha;$$

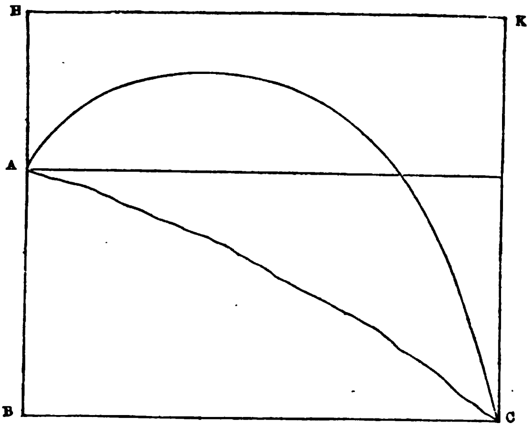
$$\therefore \text{ the required distance} = x_1 - x_2 = (1-\epsilon)\,h\sin 2\alpha.$$

$$\text{Whole time} = \frac{x_1}{V\cos\alpha} + \frac{x_2}{\epsilon V\cos\alpha} = \frac{h\sin 2\alpha}{V\cos\alpha}(1+1)$$

$$= \frac{2V}{g}\sin\alpha, \quad \text{or} \quad \left(\frac{8h\sin^2\alpha}{g}\right)^{\frac{1}{2}}.$$

52. Let A be the summit of the mountain, and C at the bottom; BA the height required; HK the directrix of the parabola described by the projectile.

$$\text{Then } AH = \frac{V^2}{2g}, \quad CK = \frac{(2V)^2}{2g};$$



$$\therefore BA = CK - AH = \frac{3V^2}{2g} = \frac{3 \times 1600^2}{2g}\left(\frac{3p}{w}\right)$$

$$= \frac{3 \times 1600^2}{2g} = \frac{3 \times 6 \cdot 5}{16 \times 13} \text{ ft.} = 2 \cdot 1181 \text{ miles.}$$

54. $$R' = 4h \frac{\sin(\alpha + \iota)\cos\alpha}{\cos^2 \iota} = 450 \operatorname{cosec} \iota,$$

and $V^2 = 1600^2 \left(\frac{3p}{196}\right)$; since a 13-inch shell weighs 196 lbs.;

$$\therefore \frac{2}{g} \times \frac{1600^2 \times 3p}{196} \frac{\sin(\alpha + \iota)\cos\alpha}{\cos^2 \iota} = 450 \operatorname{cosec} \iota;$$

$$\therefore p = \frac{75 \times 196g \operatorname{cosec} 5^\circ 30' \cos^2 5^\circ 30'}{1600^2 \sin 30^\circ \cos 24^\circ 30'} = 4 \text{lb. } 3 \text{oz.},$$

$$t = \frac{2V}{g} \frac{\sin(\alpha + \iota)}{\cos \iota} = \frac{V}{g} \sec 5^\circ 30' = 12\tfrac{2}{3} \text{ sec.}$$

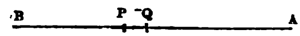
ROTATION OF BODIES.

I. Moment of Inertia.

Ex. 6.

1. Let AB be the rod, and suppose it to revolve about an axis perpendicular to it at B.

Let $AB=a$, $BP=r$, $PQ=dr$, then in this case dM is proportional to dr;

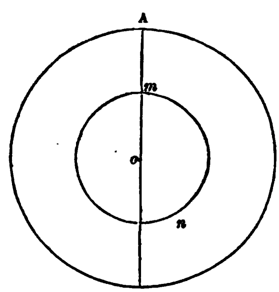


$$\therefore\ k^2=\frac{\int r^2dr}{r}=\frac{\frac{1}{3}r^3}{r}=\frac{1}{3}r^2,$$

and when $r=a$,

$$k^2=\frac{1}{3}a^2.$$

2. Let $OA=a$, $Om=r$, then the circumference $mn=2\pi r$, and dM is proportional to $2\pi rdr$,



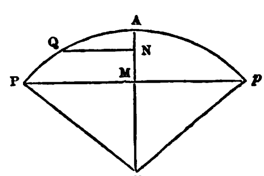
$$\text{and } k^2=\frac{\int r^2dM}{M}$$

$$=\frac{\int 2\pi r^3dr}{\int 2\pi rdr}=\frac{r^2}{2},$$

and when $r=a$,

$$k^2=\frac{1}{2}a^2.$$

3. Let $CA=a$, $NQ=r$, $AQ=s$, then $dM=2ds$, and $k^2M=2\int r^2ds$,



$$ds=\frac{adr}{(a^2-r^2)^{\frac{1}{2}}},$$

$$k^2.\text{arc } PAp=\int\frac{2ar^2dr}{(a^2-r^2)^{\frac{1}{2}}}$$

$$=C-ar\,(a^2-r^2)^{\frac{1}{2}}+a^3\sin^{-1}\frac{r}{a},$$

and this being taken between the limits $r=0$ and $r=PM=c$ suppose, we have

$$k^2 . 2\sin^{-1}\frac{c}{a} = \frac{a^2}{2} . 2\sin^{-1}\frac{c}{a} - c(a^2-c^2)^{\frac{1}{2}};$$

$$\therefore\ k^2 = \frac{a^2}{2} - \frac{c(a^2-c^2)^{\frac{1}{2}}}{2\sin^{-1}\frac{c}{a}}.$$

7. Let a, b be the semiaxes of the ellipse, and x, y the co-ordinates of any point of the curve referred to a, b as axes of co-ordinates, then for the moment of inertia about the axis of a of a quadrant of the ellipse, we have

$$Mk^2 = \int y^2 x dy = \frac{a}{b}\int y^2 (b^2-y^2)^{\frac{1}{2}} dy;$$

$$\text{and } \int y^2 (b^2-y^2)^{\frac{1}{2}} dy = \frac{1}{3}\int (b^2-y^2)^{\frac{3}{2}} dy \begin{cases} \text{from } y=b \\ \quad \text{to } y=0 \end{cases}$$

$$= \frac{1}{4} b^2 \int (b^2-y^2)^{\frac{1}{2}} dy \text{ (same limits)}$$

$$= \frac{1}{16}\pi b^4;$$

therefore for the moment of inertia of the whole ellipse we have

$$4Mk^2 = \frac{1}{4}\pi a b^3,$$

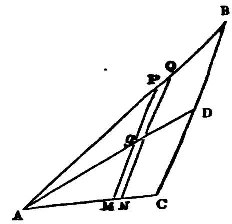
$$\text{and } 4M = \pi a b,$$

$$\text{hence } k^2 = \frac{1}{4} b^2.$$

If we denote the radius of gyration about the axis b by k', we shall have by similar reasoning

$$k'^2 = \frac{1}{4} a^2.$$

10. Let ABC be the triangle, A the angular point through which the line passes, PQ a very small part of AB. Draw PM, QN parallel to BC, and let AD bisect BC.



M = the mass of the triangle, a, b, c the sides BC, AC, AB,

$$AM = x, \quad MN = \Delta x,$$

u = moment of inertia of the triangle AMP about A,

Δu = moment of inertia of $PMNQ$ about A.

Now we may regard $PMNQ$ as a very narrow parallelogram whose centre of gravity is g the middle point of MP.

$\therefore$ Δu = mass of $PN \times Ag^2$ + its mom. of inert. about g; and mom. of inert. of PN about g = mom. of inert. of gN + mom. of inert. of gQ

$$= \frac{\text{mass } PN}{2} \times \frac{gM^2}{3} + \frac{\text{mass } PN}{2} \times \frac{gP^2}{3}$$

$$= (\text{mass of } PN)\,\frac{Mg^2}{3};$$

$$\therefore\ \Delta u = (\text{mass of } PN)\left(Ag^2 + \frac{1}{3}Mg^2\right);$$

$$\text{but mass of } PN = M\frac{\text{area of } PN}{\text{area of } ABC} = M\,\frac{\frac{ax}{b}\,.\,\Delta x\,.\sin C}{\frac{1}{2}ab\sin C}$$

$$= 2M\frac{x\Delta x}{b^2};$$

$$\text{also } Ag = AD\frac{x}{b}, \text{ and } Mg = \frac{1}{2}\cdot\frac{ax}{b};$$

$$\therefore\ \Delta u = 2M\,.\,\frac{x\Delta x}{b^2}\left(AD^2\,.\,\frac{x^2}{b^2} + \frac{a^2x^2}{12b^2}\right);$$

$$\therefore\ du = \frac{2M}{b^4}\left(AD^2 + \frac{1}{12}a^2\right)x^3dx;$$

$$\therefore\ k^2M=\int_0^b \frac{2M}{b^4}\left(AD^2+\frac{1}{12}a^2\right)x^3dx;$$

$$=\frac{2M}{b^4}\left(AD^2+\frac{1}{12}a^2\right).\frac{b^4}{4}=\frac{M}{24}(a^2+12AD^2);$$

$$\therefore\ k^2=\frac{1}{24}(a^2+12AD^2)=\frac{1}{24}(6b^2+6c^2-2a^2)$$

$$=\frac{1}{12}(3b^2+3c^2-a^2);$$

13. Divide the polygon into isosceles triangles by lines drawn from the centre to the angular points; then A being the area of one of the triangles, and R, r the radius of the circumscribed and inscribed circles, we have

$$\text{mom. of inert. of triangle}=\frac{R^2+2r^2}{6}.A;$$

$$\therefore\ k^2M=\frac{R^2+2r^2}{6}nA=\frac{R^2+2r^2}{6}M=\frac{R^2}{6}\left(1+2\cos^2\frac{\pi}{n}\right),$$

and if a be a side, we have

$$k^2M=\frac{a^2}{24}\left\{\frac{1+2\cos^2\frac{\pi}{n}}{\sin^2\frac{\pi}{n}}\right\}M=\frac{a^2}{12}\left\{\frac{2+\cos\frac{2\pi}{n}}{1-\cos\frac{2\pi}{n}}\right\}M;$$

$$\therefore\ k^2=\frac{a^2}{12}\left\{\frac{2+\cos\frac{2\pi}{n}}{1-\cos\frac{2\pi}{n}}\right\}.$$

16. Let this diameter be taken as axis of x, the centre being the origin of co-ordinates. Let x, $x+\Delta x$ be the distances of the circular faces of a thin circular slice of the sphere, at right angles to the diameter from the origin; then y being the radius of this section, its volume will $=\pi y^2\Delta x$,

$$\text{and the mass of the slice}=\frac{3M}{4a^3}y^2\Delta x,$$

$$\text{and its mom. of inert.}=\frac{3M}{4a^3}.y^2\Delta x.\frac{y^2}{2};$$

therefore mom. of inert. of the sphere

$$=\frac{3M}{8a^3}\int y^4\,dx\begin{cases}\text{from } x=-a,\\ \text{to } x=+a,\end{cases}$$

$$\text{or } k^2M = \frac{3M}{8a^3}\int(a^2-x^2)^2\,dx = \frac{3M}{8a^3}\left(a^4x - \frac{2}{3}a^2x^3 + \frac{1}{5}x^5\right)$$

$$= \frac{2a^2}{5}M;$$

$$\therefore\ k^2 = \frac{2}{5}a^2.$$

19. Let x be the distance of any thin circular slice of the cylinder from the middle point of its axis; Δx the thickness of the slice; a the radius, and $2c$ the length of the cylinder. Then the moment of inertia of the slice about any diameter being

$$= \frac{1}{4}\pi a^4\Delta x;$$

$$\text{mom. inertia about the axis of gyration} = \pi a^2\Delta x\left(x^2 + \frac{1}{4}a^2\right);$$

$$\therefore\ k^2M = \pi a^2\int\left(x^2 + \frac{1}{4}a^2\right)dx \begin{cases}\text{from } x = -c,\\ \text{to } x = +c,\end{cases}$$

$$= \pi a^2\left(\frac{2}{3}c^3 + \frac{1}{2}ca^2\right) = 2\pi a^2c\left(\frac{1}{3}c^2 + \frac{1}{4}a^2\right)$$

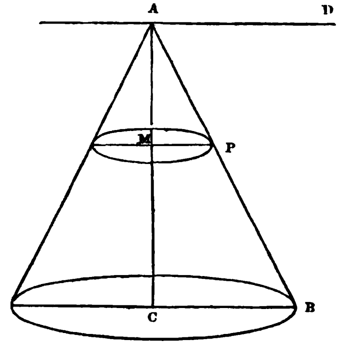
$$= \left(\frac{1}{3}c^2 + \frac{1}{4}a^2\right)M;$$

$$\therefore\ k^2 = \frac{1}{3}c^2 + \frac{1}{4}a^2;$$

21. Let CB be the base of the cone, AD the axis of gyration, MP any section,

$$AM = x,\quad MP = nx,$$

$$AC = c,\quad CB = a = nc.$$



It may easily be found that the mom. inert. of circle MP round AD

$$= \text{circle}\left(x^2 + \frac{n^2x^2}{4}\right)$$

$$= \pi n^2\left(1 + \frac{n^2}{4}\right)x^4;$$

$$\therefore\ k^2M = \pi n^2\left(1+\frac{n^2}{4}\right)\int x^4dx = \frac{\pi n^2x^5}{5}\left(1+\frac{n^2}{4}\right),$$

and, for the whole cone,

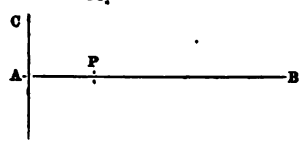
$$k^2M = \frac{\pi n^2c^5}{5}\left(1+\frac{n^2}{4}\right);$$

$$\text{but } M = \frac{\pi n^2c^2}{3}c;$$

$$\therefore\ k^2 = \frac{3c^2}{5}\left(1+\frac{n^2}{4}\right) = \frac{3}{5}\left(\frac{a^2}{4}+c^2\right) = \frac{3}{20}(a^2+4c^2).$$

23. Let AB be the rod, AC the axis of gyration.

Take any point P in AB, and let $AP=r$, $AB=l$, ρ the density at P, such that $\rho=\mu r^n$ where μ is constant,



$$\text{then } k^2M = \int \rho r^2dr = \mu\int_0^l r^{n+2}dr$$

$$= \frac{\mu l^{n+3}}{n+3}.$$

$$\text{Also } M = \int \rho dr = \mu\int r^n dr = \frac{\mu l^{n+1}}{n+1};$$

$$\text{whence } k^2 = \frac{n+1}{n+3}l^2 \quad\ldots\ldots\ldots\ldots(1).$$

Similarly, if the axis of gyration be perpendicular to the rod at B, we find

$$k'^2 = \frac{2}{(n+2)(n+3)}l^2 \quad\ldots\ldots\ldots\ldots(2);$$

therefore from (1) and (2) we get

$$k^2 = \frac{1}{2}(n+1)(n+2)k'^2,$$

and if $k=6k'$, we have

$$n^2+3n+2=72,$$

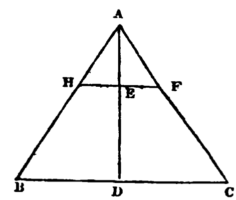
$$\text{or } n^2+3n=70;$$

$$\text{whence } n=7 \text{ or } -10.$$

II. Centre of Oscillation.

Ex. 7.

1. Let ABC be the isosceles triangle, A being the point of suspension. Draw AD perpendicular to BC.



Let $AE = x$,

$\angle BAC = \theta$,

A = area of ΔFHA,

and let l be the length of the perfect pendulum, then

$$l = \frac{\int x^2 dA}{\int x dA} \quad \text{.............................} \quad (1),$$

but $A = x^2 \tan \frac{1}{2}\theta$;

$$\therefore\ dA = 2 \tan \frac{1}{2}\theta \,.\, x dx;$$

therefore, substituting in (1), we have

$$l = \frac{\int x^3 dx}{\int x^2 dx} = \frac{3}{4} x;$$

and for the whole triangle, if $AD = a$, we have

$$l = \frac{3}{4} a;$$

therefore, if L be the length of the second's pendulum, the time of oscillation is

$$\left(\frac{l}{L}\right)^{\frac{1}{2}} = \left(\frac{3a}{4L}\right)^{\frac{1}{2}}.$$

4. Generally, if k denote the radius of gyration, h the distance between the centre of gravity and point of suspension, and l the length of the simple pendulum; then

$$l = \frac{k^2}{h} \quad \text{.............................} \quad (1),$$

and by the rules for the centre of gravity,

$$h = r - \frac{\text{rad.} \times \text{chord}}{\text{arc}} = r - \frac{r \times r}{\frac{1}{3}\pi r} = r\left(1 - \frac{3}{\pi}\right).$$

Since the arc contains 60°.

Also by Ex. 6, (5) (Mom. Inert.), we find

$$k^2 = 2r^2\left(1 - \frac{3}{\pi}\right);$$

therefore, substituting in (1), we have

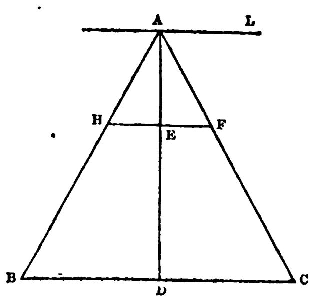
$$l = \frac{2r^2\left(1 - \frac{3}{\pi}\right)}{r\left(1 - \frac{3}{\pi}\right)} = 2r.$$

Hence, L being the length of the second's pendulum, the time of an oscillation is

$$t = \left(\frac{2r}{L}\right)^{\frac{1}{2}},$$

and is independent of the length of the arc.

6. Let ABC be a section of the pyramid through its vertex perpendicular to its base, AL the given axis.



Let $AE = x$, $EF = nx$;

$$AD = h, \quad DC = \frac{a}{2} = nh.$$

Now the moment of inertia of the square of which HF is a transverse section, round AL

$$= \text{square} \times \left(x^2 + \frac{n^2x^2}{3}\right)$$

$$= n^2\left(1 + \frac{n^2}{3}\right)x^4.$$

Therefore for the moment of inertia of the pyramid AHF round AL, we have

$$k^2M = n^2\left(1 + \frac{n^2}{3}\right)\int x^4 dx = \frac{n^2x^5}{5}\left(1 + \frac{n^2}{3}\right),$$

and for the whole pyramid, we have

$$k^2M = \frac{n^2h^5}{5}\left(1 + \frac{n^2}{3}\right),$$

and since $M = \frac{1}{3}n^2h^2 \times h$; therefore,

$$k^2 = \frac{3}{5}h^2\left(1 + \frac{n^2}{3}\right) = \frac{3}{5}h^2\left(1 + \frac{a^2}{12h^2}\right) = \frac{12h^2 + a^2}{20},$$

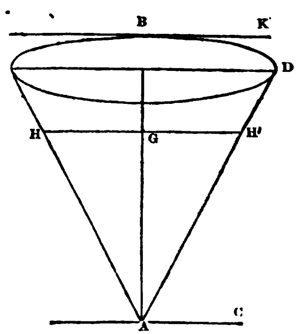
and the distance of the centre of gravity from the point of suspension is $\frac{3}{4}h$; therefore if l be the length of the perfect pendulum, we have

$$l = \frac{k^3}{\frac{3}{4}h} = \frac{12h^2 + a^2}{15h};$$

therefore the time of an oscillation is

$$t = \left(\frac{12h^2 + a^2}{15hL}\right)^{\frac{1}{2}}.$$

9. Let ADB be the cone, G its centre of gravity, BK the given axis which is a tangent to the circumference of the base of the cone at B.



Draw AC, GH parallel to BK, and therefore perpendicular to the axis of the cone. Let c = alt. of the cone, and a the radius of its base.

Now,

$$\text{mom. inert. round } AC = \text{mom. inert. round } GH + Aa^2 \times M,$$

and,

$$\text{mom. inert. round } BK = \text{mom. inert. round } GH + GB^2 \times M;$$

$\therefore$ mom. inert. round BK

$$= \text{mom. inert. round } AC + (BG^2 - AG^2)\, M,$$

and by Ex. 6, 21, moment of inertia, we have

$$\text{mom. inert. round } AC = \frac{3}{5}\left(c^2 + \frac{a^2}{4}\right) M.$$

$$\text{Also, } AG^2 = \frac{9c^2}{16}; \quad GB^2 = \frac{c^2}{16} + a^2;$$

$$\therefore \text{mom. inert. round } BK = \frac{23a^2 + 2c^2}{20} M,$$

and if l be the length of the simple pendulum, we have

$$l = \frac{\text{mom. inert. round } BK}{GB \times M}$$

$$= \frac{23a^2 + 2c^2}{20} \div \frac{(16a^2 + c^2)^{\frac{1}{2}}}{4}.$$

Therefore the time of an oscillation is

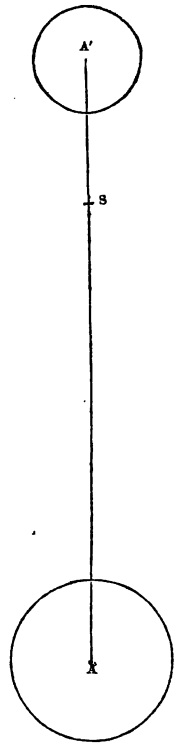
$$t = \left\{\frac{23a^2 + 2c^2}{5\,(16a^2 + c^2)^{\frac{1}{2}}\, L}\right\}^{\frac{1}{2}}.$$

13. Let m, $2m$ be the masses of the particles suspended at the distances of 30 and x inches respectively from a horizontal axis; and let L = length of the second's pendulum = 39·1393 inches.

$$\text{Then } L = \frac{m \times 30^2 + 2mx^2}{m \times 30 + 2mx} = \frac{x^2 + 15 \times 30}{x + 15};$$

$$\therefore x^2 - Lx = 15\,(L - 30).$$

Hence $x = 42\frac{3}{8}$ in. nearly.



15. Let t be the time of an oscillation about S,

l be the length of the corresponding simple pendulum,

and L be the length of the second's pendulum.

$$\text{Then } t=\pi\sqrt{\frac{l}{g}},$$

$$1=\pi\sqrt{\frac{L}{g}};$$

$$\therefore\ t=\left(\frac{l}{L}\right)^{\frac{1}{2}}.$$

$$\text{Now } l=\frac{\text{moment of inertia about } S}{\text{moment of the mass about } S}$$

$$=\frac{5^3\left(40^2+\frac{2}{5}\times 5^2\right)+3^3\left(12^2+\frac{2}{5}\times 3^2\right)}{5^3\times 40-3^3\times 12},$$

which reduced and substituted in the above expression for t, gives the time of an oscillation = 1·0589 sec.

17. Let m denote the mass of the bob of the perfect pendulum, and l its length; n the mass of the given weight, and x the distance of its point of attachment from the centre of suspension: d the distance between the centre of suspension and the centre of oscillation of the compound pendulum. Then we shall have, m and n being both of indefinitely small volume,

$$d=\frac{ml^2+nx^2}{ml+nx}.$$

Now the shorter the rod of a simple pendulum, the shorter will be the time of its oscillations; hence we must have d a minimum; therefore differentiating with respect to x we get

$$\frac{2nx\,(ml+nx)-n\,(ml^2+nx^2)}{(ml+nx)^2}=0,$$

$$\text{hence } nx^2+2mlx=ml^2,$$

$$\text{whence, } x=\frac{l}{n}\left\{(m^2+mn)^{\frac{1}{2}}-m\right\}.$$

19. Let K be the radius of gyration about the horizontal tangent,

and K' be the radius of gyration about the horizontal axis at right angles to the former.

$$\text{Then } K^2 = \frac{1}{4}a^2 + c^2 = \frac{5}{4}a^2, \text{ Ex. 6, 6,}$$

$$K'^2 = \frac{1}{2}a^2 + a^2 = \frac{3}{2}a^2; \quad \text{Ex. 6, 2;}$$

$$\therefore \frac{t}{t'} = \left(\frac{l}{l'}\right)^{\frac{1}{2}} = \frac{K}{K'} = \left(\frac{5}{6}\right)^{\frac{1}{2}}.$$

21. Let k^2M = the mom. inert. of the sector about the given axis; h the distance of its centre of gravity from the centre; a its radius, and θ the angle at the centre. Then

$$k^2M = \int \frac{a^4\,d\theta}{4} = \frac{\theta \times a^4}{4},$$

$$\text{but } M = \frac{a^2\theta}{2}, \text{ and } h = \frac{2}{3}\cdot\frac{c}{\theta},$$

(vide Ex. 6, 8, Statics, c being the chord of the arc); therefore the length of the simple pendulum

$$= \frac{k^2}{h} = \frac{3a^2\theta}{4c} = \frac{a\theta}{2} \text{ by question;}$$

$$\therefore \sin\frac{1}{2}\theta = \frac{c}{2a} = \frac{3}{4};$$

$$\therefore \theta = 2\sin^{-1}\left(\frac{3}{4}\right).$$

25. Let a = radius of cylinder, $2c$ = the length; then if K be the radius of gyration about the axis through one extremity of the cylinder, we have (Ex. 6, 19),

$$k^2 = \frac{1}{4}a^2 + \frac{1}{3}c^2 + c^2 = \frac{1}{4}a^2 + \frac{4}{3}c^2.$$

$$\text{Hence } L = \frac{\text{moment of inertia}}{\text{moment of mass}} = \frac{\frac{1}{4}a^2 + \frac{4}{3}c^2}{c}.$$

Let x be the distance from the upper extremity of the axis about which the cylinder would oscillate once in n seconds, and l the length of the corresponding simple pendulum.

$$\text{Then } n=\pi\sqrt{\frac{l}{g}},\quad 1=\pi\sqrt{\frac{L}{g}};\quad \therefore n=\left(\frac{l}{L}\right)^{\frac{1}{2}},$$

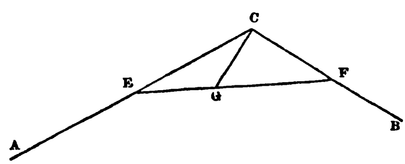
$$\text{and } \therefore l=n^2\left(\frac{a^2}{4c}+\frac{4c}{3}\right).$$

$$\text{But } l=\frac{\left(\frac{1}{4}a^2+\frac{1}{3}c^2\right)+(c-x)^2}{c-x};$$

$$\therefore x^2-\left\{2c-n^2\left(\frac{a^2}{4c}+\frac{4c}{3}\right)\right\}x=(n^2-1)\left(\frac{a^2}{4}+\frac{4c^2}{3}\right),$$

an equation which determines x.

28. Let $CA=a$,
$CB=b$,
$\angle ACB=\theta$;



bisect CA in E, and CB in F; join EF, and let G be the centre of gravity of the lever.

Then by Ex. 5, 19, Statics,

$$CG=\frac{(a^4+b^4+2a^2b^2\cos\theta)^{\frac{1}{2}}}{2(a+b)}.$$

$$\text{Hence } l=\frac{a\left(\frac{1}{3}a^2\right)+b\left(\frac{1}{3}b^2\right)}{(a+b)\,CG}=\frac{2}{3}\,\frac{a^3+b^3}{(a^4+b^4+2a^2b^2\cos\theta)^{\frac{1}{2}}}.$$

30. The general value of the tension of the string, or, which is the same thing, of the reaction of the curve (instead of the tension) against a particle moving along the curve is given by the formula

$$R=Y\frac{dx}{ds}-X\frac{dy}{ds}\pm\frac{1}{\rho}.\frac{ds^2}{dt^2}\text{..................... (1)},$$

where X, Y represent the resolved parts of the accelerating force

acting on the particle, parallel to the axes of co-ordinates; and ρ denotes the radius of curvature of the curve: the positive or negative sign is to be taken according as the particle is moving on the concave or the convex side of the curve.

Now the body oscillates in the cycloid by the action of gravity; and the resolved force of gravity along the normal in a direction opposite to the reaction is $2g \cdot \frac{dy}{ds}$; and therefore by (1), the particle moving on the concave side of the curve,

$$R = 3g\frac{dy}{ds} + \frac{1}{\rho} \cdot \frac{ds^2}{dt^2} \qquad\qquad (2).$$

$$\text{Now } \quad dy = \left(\frac{2a - x}{x}\right)^{\frac{1}{2}} dx; \quad ds = \left(\frac{2a}{x}\right)^{\frac{1}{2}} dx;$$

$$\therefore \frac{dy}{ds} = \left(\frac{2a - x}{2a}\right)^{\frac{1}{2}},$$

and the radius of curvature at any point of the cycloid is

$$\rho = 2\{2a(2a - x)\}^{\frac{1}{2}}.$$

Also if h be the altitude due to the initial velocity of the particle

$$\frac{ds^2}{dt^2} = 6g(h - x).$$

Therefore substituting in (2) we have,

$$R = 3g\left(\frac{2a - x}{2a}\right)^{\frac{1}{2}} + 3g\frac{h - x}{\{2a(2a - x)\}^{\frac{1}{2}}}$$

$$= 3g\frac{2a + h - 2x}{\{2a(2a - x)\}^{\frac{1}{2}}}.$$

But by the conditions of the problem

$$h = 2a;$$

$$\therefore R = 6g\left\{\frac{2a - x}{2a}\right\}^{\frac{1}{2}} \qquad\qquad (3).$$

Now if θ be the angle which the string makes with the axis of x, in any position, we have

$$\sin\theta = \frac{dx}{ds} = \left(\frac{x}{2a}\right)^{\frac{1}{2}};$$

$$\therefore \cos\theta = \frac{dy}{ds} = \left(\frac{2a-x}{2a}\right)^{\frac{1}{2}};$$

$$\therefore R = 6g\cos\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(4).$$

When the body oscillates in a semicircle, we have

$$dy = \frac{(r-x)\,dx}{(2rx-x^2)^{\frac{1}{2}}}; \quad ds = \frac{rdx}{(2rx-x^2)^{\frac{1}{2}}};$$

$$\therefore \frac{dy}{ds} = \frac{r-x}{r},$$

$$\text{and } \rho = r,$$

and since r is the altitude from which the body begins to fall,

$$\frac{ds^2}{dt^2} = 4g\,(r-x);$$

$$\therefore R = 2g\frac{r-x}{r} + \frac{4g\,(r-x)}{r} = 6g\frac{r-x}{r};$$

and if θ be the angle which the string makes with the axis of x, we have

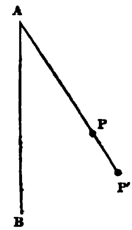
$$R = 6g\cos\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(5).$$

Equations (4) and (5) shew the truth of the proposition.

III. D'Alembert's Principle.

Ex. 8.

1. Draw AB vertically downwards, and let $\angle PAB = \theta$; also let ds, ds', denote the elements of the circular paths described by P, P', in a small time dt, estimated in a direction corresponding to an increase of θ. Then the effective moving forces of the particles PP', are



$$P\frac{d^2s}{dt^2},\quad P'\frac{d^2s'}{dt^2},$$

the moments of which about the point A are

$$P\times AP\frac{d^2s}{dt^2},\quad P'\times AP'\frac{d^2s'}{dt^2}.$$

Also the moments of the impressed forces are

$$-P\times AP\times g\sin\theta,\quad -P'\times AP'\times g\sin\theta.$$

Hence by D'Alembert's Principle

$$P\times AP\frac{d^2s}{dt^2}+P'\times AP'\frac{d^2s'}{dt^2}+(P\times AP+P'\times AP')\,g\sin\theta=0.$$

$$\text{Now } ds=AP.d\theta,\quad ds'=AP'.d\theta;$$

$$\therefore\ (P\times AP^2+P'\times AP'^2)\frac{d^2\theta}{dt^2}+(P\times AP+P'\times AP')\,g\sin\theta=0.$$

Hence it appears that the rod will oscillate isochronously with a perfect pendulum,

$$l=\frac{P\times AP^2+P'\times AP'^2}{P\times AP+P'\times AP'}.$$

2. Let m, m' be the masses of the two particles; α, α' the inclinations of the planes to the horizon; and x, x' the distances of the particles from the common summit of the planes at any time. Then the impressed accelerating forces on the particles m, m' estimated down the two planes will be $g\sin\alpha$, $g\sin\alpha'$,

respectively; and the effective accelerating forces, estimated in the same directions, will be $g\frac{d^2x}{dt^2}$, $g\frac{d^2x'}{dt^2}$. Hence, by D'Alembert's Principle, we have

$$g(m\sin\alpha - m'\sin\alpha') = m\frac{d^2x}{dt^2} - m'\frac{d^2x'}{dt^2} \dots\dots\dots\dots (1).$$

If a denote the length of the thread, then

$$x + x' = a; \quad \therefore \frac{d^2x}{dt^2} + \frac{d^2x'}{dt^2} = 0;$$

hence, from (1),

$$(m+m')\frac{d^2x}{dt^2} = g(m\sin\alpha - m'\sin\alpha') \dots\dots\dots\dots (2),$$

which determines the accelerating force on the two particles.

Let T denote the tension of the thread, then for the equilibrium of the impressed moving forces T, $mg\sin\alpha$, exerted on the particle m, and the effective moving force $m\frac{d^2x}{dt^2}$ applied in a direction opposite to its own, we shall have

$$T = m\left(g\sin\alpha - \frac{d^2x}{dt^2}\right) \dots\dots\dots\dots (3),$$

and from (2) and (3) we have

$$\text{tension} = \frac{mm'g}{m+m'}(\sin\alpha + \sin\alpha').$$

4. Let a be the radius of the pulley, Mk^2 its moment of inertia; m, m' the masses of the bodies, and x the effective accelerating force on the circumference of the pulley A, which is likewise the accelerating force on P downwards, and on Q upwards. Let Tg be the tension of the string AP, and $T'g$ that of the string BQ. Therefore the force impressed on the circumference of the pulley is

$$Tg - T'g,$$

and therefore

$$x = \frac{(T-T')ga^2}{Mk^2} \dots\dots\dots\dots (1).$$

But the accelerating force on P is

$$x = \frac{(m - T)\,g}{m},$$

and the accelerating force on Q is

$$x = \frac{T' - m'}{m'} \cdot g;$$

$$\therefore\ mx = (m - T)\,g, \quad m'x = (T' - m')\,g;$$

$$\therefore\ (m + m')\,x = (T' - T)\,g + (m - m')\,g \ldots\ldots(2).$$

And from equations (1) and (2) we get

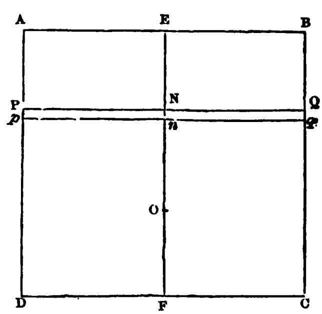
$$Mk^2x + (m + m')\,a^2x = (m - m')\,ga^2;$$

$$\text{therefore accelerating force} = x = \frac{(m - m')\,ga^2}{(m + m')\,a^2 + Mk^2};$$

$$\text{therefore tension of } AP = m\,(g - x) = \frac{(2m'a^2 + Mk^2)\,gm}{(m + m')\,a^2 + Mk^2},$$

$$\text{and tension of } BQ = m'\,(x + g) = \frac{(2ma^2 + Mk^2)\,m'g}{(m + m')\,a^2 + Mk^2}.$$

5. Let the square be moveable about AB as an axis: the point required is the centre of percussion. Let EF bisect the opposite sides AB, CD; then with regard to EF, the parts AF, FB are symmetrical; and therefore the centre of percussion (O) will be in the line EF, and be determined by the equation



$$EO = \frac{\text{moment of inertia about } AB}{\text{moment of mass } \ldots\ldots\ldots\ldots}.$$

Let $EN = x$, $Nn = h$, $AB = a = nh$, $M =$ mass of AC.

$$\text{Then } \frac{\text{mass of } Pq}{\text{mass of } AC} = \frac{ah}{a^2};$$

$$\text{therefore moment of inertia of } Pq \text{ about } AB = M \times \frac{h}{a}x^2;$$

therefore moment of inertia of AC

$$= M\frac{h}{a}\{h^2 + (2h)^2 + (3h)^2 + \ldots + (nh)^2\}$$

$$= M\frac{h^3}{a}\frac{n(n+1)(2n+1)}{6} = M\frac{(nh)^3}{3a}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)$$

$$= M\frac{a^2}{3}, \text{ when } n \text{ is increased indefinitely.}$$

$$\text{Moment of mass} = M\frac{a}{2};$$

$$\therefore\ EO = \frac{2}{3}a.$$

6. For the cylinder revolving about its own axis the moment of inertia is $\frac{1}{2}Mr^2$, Ex. 6, 18,

where M measures the weight of the cylinder = 100 lb.,

and r is the radius of cylinder.

If f be the accelerating force on the weight $P = 15$ lb.,

$$\text{then } f = \frac{\text{moving force}}{\text{mass moved}} = \frac{Pg}{P + M\frac{k^2}{r^2}} = \frac{15g}{15+50} = \frac{3}{13}g.$$

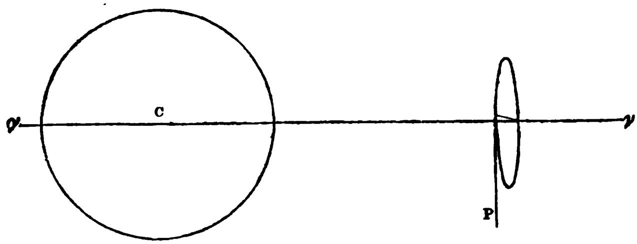
$$\text{Space required} = \frac{1}{2}ft^2 = \frac{1}{2}\left(\frac{3}{13}g\right)5^2 = 92{\cdot}856 \text{ ft.},$$

$$\text{where } g = 32{\cdot}19084 \text{ ft.}$$

8. The accelerating force

$$f = \frac{\text{moving force}}{\text{mass moved}} = \frac{Pg}{P + M\frac{k^2}{a^2}}$$

$$= \frac{20g}{20 + \frac{2}{5}(500)\frac{3^2}{\left(\frac{1}{2}\right)^2}} = \frac{g}{361};$$



since for the sphere $Mk^2 = \frac{2}{5} \times (\text{radius})^2 \times M$; Ex. 6, 16,

$$t = \left(\frac{2s}{f}\right)^{\frac{1}{2}} = 10\left(\frac{361}{g}\right)^{\frac{1}{2}} = 33{\cdot}488 \text{ sec.}$$

$$v = \sqrt{2fs} = 10\left(\frac{g}{361}\right)^{\frac{1}{2}} = 2{\cdot}986 \text{ ft.}$$

10. Let w = the weight of the pulley; then, considering the pulley as a small cylinder, its moment of inertia $= \frac{a^2}{2} w$.

Suppose P to draw Q over the pulley, then the effective accelerating force on P downwards, or on Q upwards is (as found in Ex. 4),

$$f = \frac{(P-Q)\,a^2 g}{(P+Q)\,a^2 + Mk^2} = \frac{(P-Q)\,g}{P+Q+\frac{1}{2}w} = \frac{16}{67}g;$$

$$\text{therefore time} = \left(\frac{2s}{f}\right)^{\frac{1}{2}} = \left(\frac{60 \times 67}{16g}\right)^{\frac{1}{2}} = 2{\cdot}79 \text{ sec.}$$

12. Let P and Q be the weights, P drawing up Q. Let a, b be the radii of the cylinders, and Mk^2 the moment of inertia of the machine about its axis. We shall then have impressed forces, Pg at distance a, $-Qg$ at distance b; of which the moment is $Pga - Qgb$, hence we have

$$\text{accelerating force on } P = \frac{(Pa - Qb)\,ga}{Pa^2 + Qb^2 + Mk^2},$$

$$\text{accelerating force on } Q = \frac{(Pa - Qb)\,gb}{Pa^2 + Qb^2 + Mk^2},$$

and from the data $P = 100$, $Q = 500$, $b = \frac{1}{2}$, $k = 3$, $M = 80$, $t = 5$, and $s = 10$;

$$\therefore f = \text{the force with which } Q \text{ ascends}$$

$$= \frac{(50a - 125)\,g}{100a^2 + 125 + 720}, \text{ and } \left(\text{since } s = \frac{1}{2}ft^2\right),$$

$$\text{we have } \frac{\frac{1}{2} \times (50a - 125) \times 5^2}{100a^2 + 845} = 10,$$

$$\text{or } a^2 - 20a + 58 = 0, \text{ whence } a = 16{\cdot}4 \text{ ft.}$$

Now if M', M'' be the masses of the wheel and axle respectively, then

$$\left(\frac{\frac{1}{2}M'a^2 + \frac{1}{2}M''b^2}{M}\right)^{\frac{1}{2}}$$

is the expression for the distance of the centre of gyration of the *machine* from the axis;

$$\therefore \frac{\frac{1}{2}M'a^2 + \frac{1}{2}M''b^2}{M} = \frac{134{\cdot}48M' + \frac{1}{8}M''}{80} = 3^2,$$

$$\text{and } M' + M'' = 80,$$

$$\text{hence } M' = 5\tfrac{1}{3} \text{ lb. nearly.}$$

Let T, T' be the tensions of the ropes by which P, Q respectively act, then

$$T = \frac{P(Qb^2 + Qab + Mk^2)}{Pa^2 + Qb^2 + Mk^2}, \quad T' = \frac{Q(Pa^2 + Pab + Mk^2)}{Pa^2 + Qb^2 + Mk^2}.$$

Pressure upon the axis $= T + T' + M$

$$= \frac{PQ(a+b)^2 + (P+Q)Mk^2}{Pa^2 + Qb^2 + Mk^2} + M$$

$$= \frac{50000\,(16{\cdot}9)^2 + 600 \times 80\,(3^2)}{100\,(16{\cdot}4)^2 + 500 \times {\cdot}25 + 720} + 80$$

$$= 610{\cdot}4 \text{ lb. nearly.}$$

HYDROSTATICS.

Pressure on Surfaces.

Ex. 1.

1. Let ABC be the equilateral triangle, A being in the surface, and AB vertical; g', g'', g''' the centres of gravity of AC, AB, BC respectively. Through g''' draw $g'''a$ perpendicular to the surface of the fluid; $g'''a$ will pass through g'.

Now the sides pressed being equal in length, the pressures will be proportional to the depths of their centres of gravity;

$$\therefore \text{press. on } AC : \text{press. on } AB : \text{press. on } BC = ag' : Ag'' : ag''',$$

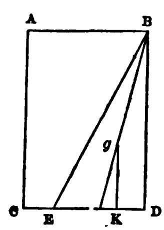
$$\text{but } Ag'' = 2ag', \quad ag''' = 3ag';$$

$$\therefore \text{press. on } AC : \text{press. on } AB : \text{press. on } BC = 1 : 2 : 3.$$

2. The surfaces pressed being equal, the pressures will be to each other as the depths of the centres of gravity of the triangles below the surface of the fluid. Now if a be the perpendicular from the vertex on the base, the depth of the centre of gravity in the first position is $\frac{2}{3}a$, and in the second $\frac{1}{3}a$;

$$\therefore \text{ratio of pressures} = \frac{2}{3}a : \frac{1}{3}a = 2 : 1.$$

7. Let $ABCD$ be the rectangle immersed in fluid, BE the line dividing it as required, g the centre of gravity of the triangle BED.



Let $ED = x$; draw gK parallel to BD; then $gK = \frac{1}{3}BD$; therefore the depth of g below the surface $= \frac{2}{3}BD$, and the area of the triangle $= 7x$;

$$\therefore \text{pressure on the triangle} = 7x \times \frac{28}{3} \times \rho,$$

and press. on remaining portion $= 9 \times 14 \times 7\rho - \dfrac{7x \times 28 \times \rho}{3}$;

$$\therefore \text{ by hypothesis } \rho\left(9 \cdot 14 \cdot 7 - \frac{7x \times 28}{3}\right) : \frac{7x \times 28}{3}\rho = 5 : 3,$$

$$\text{or } 9 - \frac{2x}{3} : \frac{2x}{3} = 5 : 3;$$

$$\therefore 9 : \frac{2x}{3} = 8 : 3;$$

$$\therefore \frac{2x}{3} = \frac{27}{8},$$

$$x = \frac{81}{16} = 5\tfrac{1}{16}.$$

9. The areas of the triangles pressed being equal, the pressures on them will be proportional to the depths of their centres of gravity below the surface of the fluid.

Now depth of centre of gravity of $\Delta\, BOA = \dfrac{1}{6} BC$,

.... $\Delta\, BOC = \dfrac{1}{2} BC$,

.... $\Delta\, DOC = \dfrac{5}{6} BC$;

$$\therefore L : M : N = \frac{1}{6} BC : \frac{1}{2} BC : \frac{5}{6} BC$$

$$= 1 : 3 : 5.$$

12. Let a = the axis, and b the double ordinate of the parabola; then the area of the parabola is $\dfrac{2}{3}$ (the area of the circumscribing rectangle) $= \dfrac{2}{3} ab$ and the depth of its centre of gravity $= \dfrac{3}{5} a$;

$$\therefore \text{ pressure on the parabola} = \frac{2}{5} a^2 b\rho,$$

similarly we find the pressure on the rectangle

$$=ab \times \frac{1}{2} a \times \rho = \frac{1}{2} a^2 b\rho;$$

$\therefore$ pressure on parabola : press. on rectangle

$$=\frac{2}{5} a^2 b\rho \ : \ \frac{1}{2} a^2 b\rho = 4 : 5.$$

13. Let h be the altitude of the given parabola, and h' that of the parabola cut off; also let c be the double ordinate of the original parabola, and that of the parabola cut off will be $c\left(\frac{h'}{h}\right)^{\frac{1}{2}}$.

The pressures on the two parabolas will be

$$\frac{2}{3} hc . \frac{3}{5} h . \rho = \frac{2}{5} h^2 c\rho,$$

$$\text{and } \frac{2}{3} h'c . \left(\frac{h'}{h}\right)^{\frac{1}{2}} . \frac{3}{5} h' . \rho = \frac{2}{5} \frac{h'^{\frac{5}{2}}}{h^{\frac{1}{2}}} . c\rho;$$

therefore the pressure on the lower portion of the proposed parabola will be

$$\frac{2}{5} c\rho \left(h^2 - \frac{h'^{\frac{5}{2}}}{h^{\frac{1}{2}}}\right),$$

hence, by the hypothesis,

$$\frac{h'^{\frac{5}{2}}}{h^{\frac{1}{2}}} : h^2 - \frac{h'^{\frac{5}{2}}}{h^{\frac{1}{2}}} = m : n;$$

$$\therefore h'^{\frac{5}{2}} : h^{\frac{5}{2}} = m : m + n,$$

$$\text{whence } h' = \left(\frac{m}{m+n}\right)^{\frac{2}{5}} h.$$

17. Let the edge of the cubical vessel $= 2a$,

ρ, σ be the densities of water and mercury respectively.

Then pressure on upper half of a side $= 2a . a . \frac{a}{2} \rho = a^3 \rho$.

To find the pressure on the lower half we must first find the thickness (z) of a lamina of mercury, the weight of which is equal to the weight of water: then

$$z\sigma = a\rho; \quad \therefore\ z = a\frac{\rho}{\sigma}.$$

$$\text{Pressure on lower half} = 2a \,.\, a\left(a\frac{\rho}{\sigma} + \frac{a}{2}\right)\sigma;$$

$$\text{therefore pressure on a side} = a^3(\rho + 2\rho + \sigma),$$

$$\text{pressure on base} = 4a^2(a\rho + a\sigma).$$

Hence, pressure on sides : pressure on base $= \sigma + 3\rho : \sigma + \rho$.

18. The area of each of the equal sides of the pyramid is $22{\cdot}561 \times 5$; and the area of the base is 100.

The depth of the centre of gravity of each of the sides is $\frac{44}{3}$;

$$\text{therefore pressure on the bottom} = 100 \times 22 \times \rho,$$

$$\text{pressure on the side} = 112{\cdot}805 \times \frac{44}{3} \times \rho,$$

$$\text{the weight of the water} = \frac{1}{3} \times 100 \times 22 \times \rho = \frac{2200}{3}\rho;$$

therefore press. on base : press. on each side : weight of water

$$= 100 \times 22 : 22{\cdot}561 \times 5 \times \frac{44}{3} : \frac{2200}{3}$$

$$= 3 : 2{\cdot}256 : 1.$$

21. If ρ be the density of water, and r the radius of the sphere; then

$$\text{pressure on the sphere} = 4\pi r^3\rho,$$

$$\text{weight of mercury} = \frac{4}{3}\pi r^3 \times 13{\cdot}568\rho;$$

$$\text{therefore pressure : weight} = 1 : \frac{13{\cdot}568}{3}$$

$$= 1 : 4{\cdot}523.$$

24. Let h be the height of the upper segment cut off by the dividing line, r the radius of the sphere, and ρ the density of the fluid; then,

$$\text{the surface pressed} = 2\pi hr,$$

$$\text{and the depth of the centre of gravity} = \frac{1}{2}h$$

(vide Ex. 6, 35, of Statics);

$$\text{therefore pressure on upper segment} = \pi rh^2\rho,$$

$$\text{but pressure on whole sphere} = 4\pi r^3\rho;$$

$$\text{therefore pressure on the lower segment} = \pi r\rho\,(4r^2 - h^2),$$

and, by the hypothesis,

$$4r^2 - h^2 = h^2;$$

$$\therefore\ h = r \times 2^{\frac{1}{2}}.$$

25. Let z_1, z_2, z_3 be the depths of the dividing planes below the surface of the fluid; ρ the density.

$$\text{Surface pressed in the upper portion} = 2\pi rz_1;$$

$$\text{depth of centre of gravity} = \frac{1}{2}z,$$

$$\text{therefore pressure on uppermost portion} = \pi rz_1^2\rho \ldots\ldots\ldots (1).$$

$$\text{Surface pressed in next portion} = 2\pi r\,(z_2 - z_1),$$

$$\text{and depth of centre of gravity} = \frac{1}{2}(z_2 + z_1);$$

$$\text{therefore pressure on next portion} = \pi r\,(z_2^2 - z_1^2)\,\rho \ldots\ldots (2).$$

Similarly the pressures on the remaining portions are found to be $\pi r\,(z_3^2 - z_2^2)\,\rho$, and $\pi r\,(r^2 - z_3^2)\,\rho$,

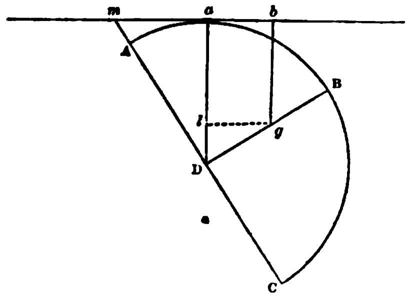
and by the hypothesis, these pressures are equal to each other,

$$\text{therefore } z_1^2 = z_2^2 - z_1^2 = z_3^2 - z_2^2 = r^2 - z_3^2,$$

$$\text{whence } z_1^2 = \frac{1}{2}z_2^2 = \frac{1}{3}z_3^2 = \frac{1}{4}r^2,$$

$$\text{whence } z_1 = \frac{1}{2}r,\quad z_2 = \frac{1}{2}r \times 2^{\frac{1}{2}},\quad z_3 = \frac{1}{2}r \times 3^{\frac{1}{2}}.$$

28. Let ABC represent the hemisphere, with its axis BD making an angle α with the vertical.



Let g be the centre of gravity of the surface of the hemisphere. From g draw gb perpendicular to the surface of the fluid, and gl to meet the vertical through D in l; then

$$gb = r - lD = r - \frac{1}{2} r \cos \alpha = \frac{r}{2}(2 - \cos \alpha),$$

the surface of the hemisphere $= 2\pi r^2$;

$$\therefore \text{ pressure required} = \pi r^3 \rho (2 - \cos \alpha).$$

33. Let z_1, z_2, z_3, z_4 be the depths of the sections, and h the height of the cylinder.

The surface pressed in the first annulus $= 14\pi z_1$,

and the depth of the centre of gravity $= \frac{1}{2} z_1$;

$\therefore$ pressure on first annulus $= 7\pi\rho z_1^2$...............(1).

Surface pressed in the second annulus $= 14\pi(z_2 - z_1)$,

and the depth of the centre of gravity $= \frac{1}{2}(z_2 + z_1)$;

$\therefore$ pressure on the second annulus $= 7\pi\rho(z_2^2 - z_1^2)$.........(2).

Similarly, pressure on the third annulus $= 7\pi\rho(z_3^2 - z_2^2)$,

.... fourth .... $= 7\pi\rho(z_4^2 - z_3^2)$,

and .... fifth .... $= 7\pi\rho(h^2 - z_4^2)$,

and, by the hypothesis, these pressures are all equal to one another, and to the pressure on the base;

$$\therefore\ z_1^2 = z_2^2 - z_1^2 = z_3^2 - z_2^2 = z_4^2 - z_3^2 = h^2 - z_4^2 = 7h;$$

$$\therefore\ h^2 = 5z_1^2 = 35h;\ \ \therefore\ h = 35.$$

$$\text{Breadth of third annulus} = z_3 - z_2$$

$$= z_1(\sqrt{3} - \sqrt{2}) = 7\sqrt{5}(\sqrt{3} - \sqrt{2}) = 4{\cdot}9749 \text{ in.}$$

36. Using the same notation as in Exs. 25 and 33, the pressures are $\pi r\rho z_1^2$, $\pi r\rho(z_2^2 - z_1^2)$, $\pi r\rho(z_3^2 - z_2^2)$, $\pi r\rho(20^2 - z_3^2)$, and by hypothesis, these pressures are in Geom. Progression, having 2 for a common ratio;

$$\therefore\ 2z_1^2 = z_2^2 - z_1^2, \qquad z_2^2 = 3z_1^2,$$

$$2z_2^2 - 2z_1^2 = z_3^2 - z_2^2, \qquad z_3^2 = 7z_1^2,$$

$$20^2 - z_3^2 = 2z_3^2 - 2z_2^2,\ \ 20^2 - 7z_1^2 = 8z_1^2;$$

$$\therefore\ 15z_1^2 = 20^2,\quad z_1 = \frac{4}{3} \times (15)^{\frac{1}{2}};$$

$$\therefore\ z_2 = \frac{4}{3}(15 \times 3)^{\frac{1}{2}},\ \text{and}\ z_3 = \frac{4}{3}(15 \times 7)^{\frac{1}{2}}.$$

38. Let h = altitude of cone, l = its slant height, and r = radius of base.

$$\text{Convex surface of cone} = \pi rl,$$

$$\text{depth of its centre of gravity} = \frac{2}{3}h;$$

$$\therefore\ \text{pressure on convex surface} = \frac{2}{3}\pi rhl\rho.$$

$$\text{Also, pressure on base} = \pi r^2 \times r \times \rho;$$

$$\therefore\ \text{whole pressure} = \frac{1}{3}\pi hr(2l + 3r)\rho.$$

40. Let h = altitude of cone, l = slant height, and a = radius of base, z = depth of dividing plane.

Then, $h - z$ = altitude of cone below the dividing plane,

$$z + \frac{h-z}{3} = \frac{1}{3}(h + 2z) = \text{depth of its centre of gravity},$$

$$\text{surface of this small cone} = \pi\frac{a}{h}(h - z) \times \frac{l}{h}(h - z);$$

$\therefore$ pressure on .... $= \dfrac{\pi al}{3h^2}(h-z)^2(h+2z)\rho.$

But pressure on given cone $= \pi al \times \dfrac{1}{3} h\rho$;

$$\therefore \frac{\pi al}{3h^2}(h-z)^2(h+2z)\rho = \frac{1}{2}\left(\frac{\pi al}{3} \times h\rho\right);$$

$$\therefore (h-z)^2(h+2z) = \frac{1}{2}h^3,$$

$$\text{or } 4z^3 - 6hz^2 + h^3 = 0;$$

the real root of which equation determines the depth of the dividing plane.

42. Let h = axis of the cone, r = radius of its base, and ρ = density of the fluid. Then the pressure on the concave surface of the cone

$$= \frac{1}{3} 2\pi rh\rho (r^2+h^2)^{\frac{1}{2}},$$

and the pressure on the base of the cone

$$= \pi r^2 h\rho;$$

therefore, by the hypothesis,

$$\frac{2}{3}(r^2+h^2)^{\frac{1}{2}} : r = 4 : 3,$$

$$\text{whence } \frac{r}{h} = \frac{1}{\sqrt{3}} = \tan 30^\circ;$$

$$\therefore \text{the vertical angle} = 60^\circ.$$

The ratio of the pressures $= \dfrac{2}{3}(r^2+h^2)^{\frac{1}{2}} : r$

$$= \frac{2}{3}\left(1+\frac{h^2}{r^2}\right)^{\frac{1}{2}} : 1.$$

Since $\dfrac{h^2}{r^2}$ is always positive, $\left(1+\dfrac{h^2}{r^2}\right)^{\frac{1}{2}}$ is greater than 1; therefore the ratio of the pressures cannot for any cone be less than $\dfrac{2}{3} : 1$ or than $2 : 3$.

44. Let h be the axis of the vessel measured vertically upwards; x, y the co-ordinates of any point in the curve, then

$2y$ is the diameter of any horizontal section; πy^2 its area, and $\pi(h-x)y^2$ the volume of the superincumbent fluid, which is therefore proportional to the pressure. Hence,

$$y^2(h-x) = c, \text{ a constant quantity,}$$

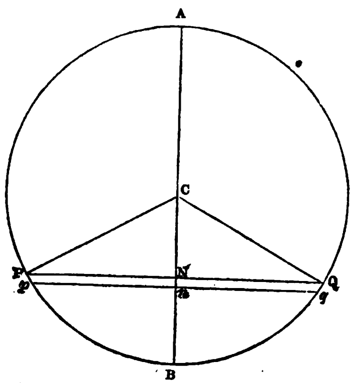
$$y^2 = \frac{c}{h-x},$$

is the equation to the curve.

Ex. 2.

1. Let a = radius of circle;

$$AN = x, \quad NP = y; \quad \angle PCQ = \theta;$$



area of $Pq = PQ \times Nn$ ultimately; let $Nn = dx$;

then, pressure on $Pq = 2xydx \times \rho$;

$\therefore$ pressure on segment $PAQ = 2\rho\int xydx$, between the limits $x = 0$ and $x = AN$.

Now $x = a + a\cos\frac{\theta}{2}$, and $y = a\sin\frac{\theta}{2}$,

$$dx = -\frac{a}{2}\sin\frac{\theta}{2}\,.\,d\theta; \quad \therefore xydx = -\frac{a^3}{2}\left(\sin^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\right)d\theta.$$

Hence,

$$2\rho\int xydx = -2\rho \times a^3 \int \left\{\frac{(1-\cos\theta)}{4}\,d\theta + \sin^2\frac{\theta}{2}\,d\,.\,\sin\frac{\theta}{2}\right\}$$

$$= C - 2\rho a^3\left(\frac{\theta - \sin\theta}{4} + \frac{1}{3}\sin^3\frac{\theta}{2}\right).$$

This integral vanishes, or the pressure $= 0$, when $\theta = 2\pi$;

$$\therefore\ C = \rho a^3 \pi;$$

$$\therefore \text{ pressure on } PAQ = a^3 \left\{\pi - \frac{\theta - \sin\theta}{2} - \frac{2}{3}\sin^3\frac{\theta}{2}\right\}\rho.$$

If PQ be the required dividing line, this pressure $= \frac{1}{2}$ pressure on the circle $= \frac{\pi a^3}{2}\rho$;

$$\therefore\ \pi - \frac{1}{2}(\theta - \sin\theta) - \frac{2}{3}\sin^3\frac{\theta}{2} = \frac{\pi}{2};$$

$$\therefore\ 4\sin^3\frac{\theta}{2} - 3\sin\theta + 3\theta = 3\pi.$$

2. Let $AP = r$,

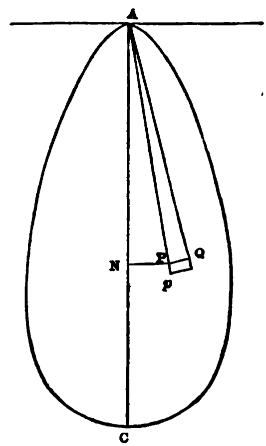
$\angle CAP = \theta$,

$AC = a$,

$Pp = dr$,

$\angle PAQ = d\theta$,

$\rho =$ density of fluid.



Elementary area

$pQ = PQ \times Pp = r d\theta \,.\, dr$ ultimately,

pressure on

$$pQ = (r d\theta\, dr) \times r\cos\theta\,\rho,$$

whole pressure on loop $= \rho \iint r^2 \cos\theta\, dr\, d\theta$

$$= \rho\int \cos\theta\, d\theta \left(\frac{1}{3}r^3 + C\right), \text{ from } r = 0 \text{ to } r = a(\cos 2\theta)^{\frac{1}{2}}$$

$$= \rho\frac{a^3}{3}\int \cos\theta\,(1 - 2\sin^2\theta)^{\frac{3}{2}}\, d\theta$$

$$= \rho\frac{a^3}{3} \times 2^{\frac{3}{2}} \int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} \cos\theta\,(c^2 - \sin^2\theta)^{\frac{3}{2}}\, d\theta,$$

if $c^2 = \frac{1}{2}$.

Let $\sin\theta = z$, then $\cos\theta\, d\theta = dz$, and

$$\int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} \cos\theta\,(c^2 - \sin^2\theta)^{\frac{3}{2}}\, d\theta = \int_{-c}^{+c} (c^2 - z^2)^{\frac{3}{2}}\, dz.$$

Now

$$\int (c^2 - z^2)^{\frac{3}{2}}\, dz = z\,(c^2 - z^2)^{\frac{3}{2}} + 3\int z^2\,(c^2 - z^2)\ \ dz$$

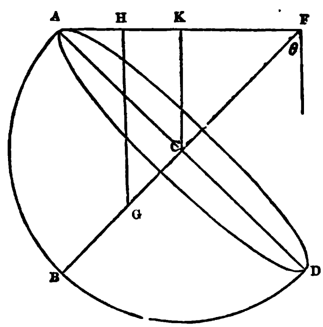
$$= \dots\dots\dots\dots - 3\int (c^2 - z^2)^{\frac{3}{2}}\, dz + 3c^2\int (c^2 - z^2)^{\frac{1}{2}}\, dz$$

$$= \frac{z}{4}(c^2 - z^2)^{\frac{3}{2}} + \frac{3c^2}{4}\int (c^2 - z^2)^{\frac{1}{2}}\, dz;$$

$$\therefore\ \int_{-c}^{+c} (c^2 - z^2)^{\frac{3}{2}}\, dz = \frac{3c^2}{4}\int_{-c}^{+c} (c^2 - z^2)^{\frac{1}{2}}\, dz = \frac{3c^2}{4} \times 2\left(\frac{\pi c^2}{4}\right) = \frac{3\pi c^4}{8}.$$

$$\text{Hence, whole pressure} = \rho\,\frac{a^3 2\sqrt{2}}{3} \times \frac{3\pi}{8\times 4} = \frac{\pi a^3 \rho}{8\sqrt{2}}.$$

4. Let θ be the inclination of the hemisphere's axis to the vertical, r the radius, ρ the density of fluid.



Then, pressure on concave surface $= 2\pi r^2 \times HG\,.\,\rho$,

$\dots\dots\dots\dots$ plane $\dots\dots\dots = \pi r^2 \times KC\,.\,\rho$.

Now $HG = FG\cos\theta = (GC + CF)\cos\theta = \left(\frac{1}{2}r + r\tan\theta\right)\cos\theta$,

and $KC = r\sin\theta$;

$\therefore$ whole pressure $= \pi\rho r^3\,(\cos\theta + 2\sin\theta + \sin\theta)$.

But $\cos\theta + 3\sin\theta$ is a maximum, when differentiating

$$-\sin\theta + 3\cos\theta = 0.$$

$$\text{Hence}\quad \tan\theta = 3, \quad \text{or}\quad \theta = \tan^{-1} 3.$$

7. Let $x =$ the depth of the chord below the surface, ρ the density at any point in the chord, and therefore $= kx^2$, k being constant;

$$\text{pressure at depth } x = \int_0^x \rho dx = k\int_0^x x^2 dx = \frac{k}{3} x^3;$$

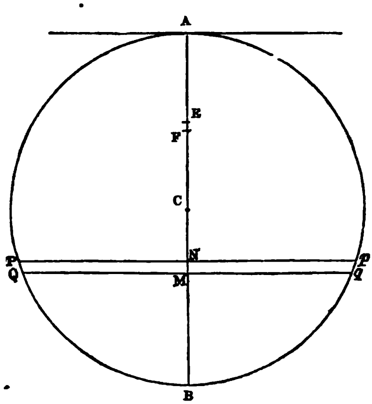
$$\text{therefore pressure on chord} = 2(a^2 - x^2)^{\frac{1}{2}} \times \frac{k}{3} x^3 \text{ a maximum};$$

$$\text{differentiating}\quad (a^2 - x^2)\, x^6 \text{ a maximum};$$

$$6a^2x^5 - 8x^7 = 0;$$

$$\therefore\ x = \left(\frac{3}{4}\right)^{\frac{1}{2}} \times \text{radius}.$$

10. Let $AB = 2r$, $AN = x$, $NM = dx$, $NP = y$,



$$\rho = \text{density at depth } r.$$

Then, area of $Pq = 2ydx$, ultimately.

Pressure on Pq = weight of fluid superincumbent on the particles in contact with Pq.

To find the weight of a column of particles AN,

$$\text{let } AE = z,\ EF = dz,\ \rho' = \text{density at } E;$$

$$\text{then } \rho' : \rho = z : r;$$

$$\therefore \text{ weight of the column } AN = \int \rho' dz = \int_0^x \frac{\rho}{r} z dz = \frac{\rho x^2}{2r};$$

$$\therefore \text{ pressure on } Pq = 2y dx \times \frac{\rho x^2}{2r} = \frac{\rho}{r}(x^2 y dx);$$

$$\therefore \text{ whole pressure on circle } = \frac{\rho}{r} \int_0^{2r} x^2 (2rx - x^2)^{\frac{1}{2}} dx$$

$$= \frac{\rho}{r} \times \frac{5\pi r^4}{8}$$

$$= \frac{5}{8} \pi r^3 \times \text{density at the centre of circle.}$$

15. Let a denote the axis of the cone, b the radius of its base, y the radius of any horizontal section, and x its depth; then

$$y = \frac{b}{a}(a - x).$$

Now let ρ be the density of the fluid at depth x; and therefore $= kx^2$, k being constant.

$$\text{Pressure at depth } x = \int_0^x \rho dx = k \int_0^x x^2 dx = \frac{k}{3} x^3.$$

$$\text{Pressure on horizontal section at depth } x = \pi y^2 \times \frac{k}{3} x^3.$$

$$\text{Hence } \frac{\pi b^2}{a^2}(a - x)^2 \times \frac{k}{3} x^3 \text{ or } (a - x)^2 x^3 \text{ is a maximum.}$$

Differentiating, we have

$$3x^2 (a - x)^2 - 2x^3 (a - x) = 0,$$

$$\therefore x = \frac{3}{5} a.$$

17. Let x = the depth of the horizontal section below the centre of the spheroid.

The section will be an ellipse, the semi-axes of which are

$$(36-x^2)^{\frac{1}{2}}, \text{ and } \frac{1}{3}(36-x^2)^{\frac{1}{2}};$$

$$\therefore \text{ area of section} = \frac{\pi}{3}(36-x^2).$$

Let ρ = density of the fluid at depth z, and therefore equal to kz where k is constant or the density at depth unity. Then

$$\text{the pressure at depth } z = \int_0^z \rho dr = \frac{k}{2}z^2.$$

$$\text{Hence, pressure on section} = \frac{\pi}{3}(36-x^2) \times \frac{k}{2}(6+x)^2,$$

and this is to be a maximum;

$$\therefore 2(36-x^2)(6+x) - 2x(6+x)^2 = 0;$$

$$\therefore 4x^2 + 12x - 72 = 0,$$

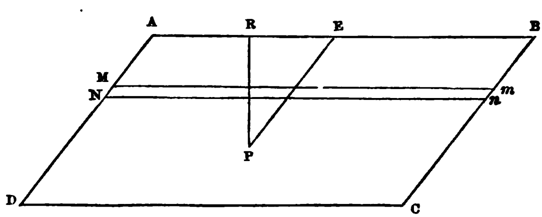
whence $x = 3$; and the depth of section $= 9$;

$$\therefore \text{ pressure} = \frac{\pi k}{6}(36-9)(6+3)^2 = \frac{1}{2}\pi k \times 9^3.$$

Centre of Pressure.

Ex. 3.

1. Let $AB = a$, $AD = b$, $\angle ABC = \alpha$;
$AM = x$, $MN = h$, $\therefore nh = b$;
Density of fluid $= \rho$.



Then area of $Mn = ah \sin \alpha$,

pressure on $Mn = ah \sin \alpha \times x \sin \alpha \,.\, \rho$ ultimately.

Moment of this pressure about $AB = ah \sin\alpha \times x \sin\alpha . \rho \times x \sin\alpha$
$$= ah\rho \sin^3\alpha \times x^2.$$

For x write h, $2h$, $3h$, ... nh successively; then sum of all such moments

$$= ah\rho \sin^3\alpha \{h^2 + (2h)^2 + (3h)^2 + \ldots\ldots + (nh)^2\}$$
$$= ah^3\rho \sin^3\alpha \frac{n(n+1)(2n+1)}{6}$$
$$= a\rho \sin^3\alpha \frac{(nh)^3}{3}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right)$$
$$= \frac{1}{3} ab^3 \sin^3\alpha . \rho,$$

since $nh = b$, and n is increased indefinitely, h being diminished indefinitely.

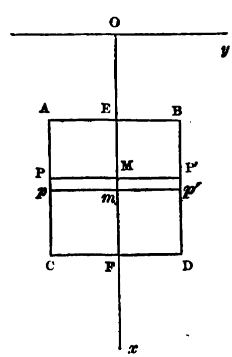
Again, area of $ABCD = ab \sin\alpha$,

$$\text{pressure on } ABCD = ab \sin\alpha \times \frac{1}{2} b \sin\alpha . \rho.$$

This pressure may be considered to act at P the centre of pressure;

$\therefore$ moment of this pressure about $AB = \frac{1}{2} ab^2 \sin^2\alpha . \rho \times RP$.

$$\text{Hence } RP \times \frac{1}{2} ab^2 \sin^2\alpha . \rho = \frac{1}{3} ab^3 \sin^3\alpha . \rho;$$
$$\therefore RP = \frac{2}{3} b \sin\alpha,$$
$$\text{or } EP = \frac{RP}{\sin\alpha} = \frac{2}{3} b.$$



2. Let $ABCD$ be the rectangular plank, FE being a straight line joining the middle points of AB, CD, and produced to meet the surface of the fluid in O. Take Ox, Oy as axes of co-ordinates. Let PP', pp' be two lines parallel to AB cutting EF in two points M, m, very near to each other, then

$$\text{element } Pp' = 2y\,dx,$$
$$\text{and its depth} = x;$$

$$\therefore \ \bar{x} \int_h^k 2yx\,dx = \int_h^k 2yx^2\,dx,$$

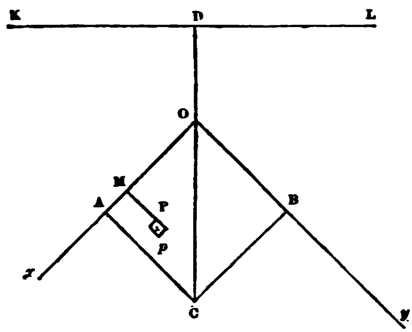
or, since y is invariable,

$$\bar{x} \int_h^k x\,dx = \int_h^k x^2\,dx;$$

$$\therefore \ \bar{x} \times \frac{1}{2}(k^2 - h^2) = \frac{1}{3}(k^3 - h^3);$$

$$\therefore \ \bar{x} = \frac{2}{3} \cdot \frac{k^3 - h^3}{k^2 - h^2}.$$

4. Let the diagonal CO produced meet the surface KDL in the point D, and let $OD = h$; and let OA, OB produced



indefinitely be the axes of x and y. Let x, y be the co-ordinates of a point P of the square, and $x + dx$, $y + dy$, of a point p indefinitely near to it; then the depth of the elementary square of which Pp is a diagonal, below the surface of the fluid, being

$$h + \frac{x + y}{\sqrt{2}},$$

and its area being $dxdy$, we have, a denoting a side of the square AB,

$$\bar{x} \int_0^a \int_0^a \left(h + \frac{x + y}{\sqrt{2}}\right) dxdy = \int_0^a \int_0^a \left(h + \frac{x + y}{\sqrt{2}}\right) xdxdy.$$

The coefficient of $\bar{x}$ in this equation is equal to

$$\int_0^a \left(ha + \frac{ax}{\sqrt{2}} + \frac{a^2}{2\sqrt{2}}\right) dx = ha^2 + \frac{a^3}{2\sqrt{2}} + \frac{a^3}{2\sqrt{2}} = \frac{a^2}{\sqrt{2}}(h\sqrt{2} + a),$$

and the right-hand side of the equation is equal to

$$\int_0^a \left(\frac{1}{2} ha^2 + \frac{a^2 y}{2\sqrt{2}} + \frac{a^3}{3\sqrt{2}}\right) dy = \frac{1}{2} ha^3 + \frac{a^4}{4\sqrt{2}} + \frac{a^4}{3\sqrt{2}} = \frac{a^3}{12\sqrt{2}}$$
$$(6h\sqrt{2} + 7a).$$

$$\text{Hence } \bar{x} = \frac{a}{12} \cdot \frac{7a + 6\sqrt{2} \, . \, h}{a + \sqrt{2} \, . \, h}.$$

(Walton's *Hydrostatical Problems*).

Otherwise, by the Integral Calculus,

$$\bar{x} = \frac{\int_0^b x^2 y dx}{\int_0^b xy dx}, \quad \bar{y} = \frac{1}{2} \cdot \frac{\int_0^b xy^2 dx}{\int_0^b xy dx},$$

$$\text{for the hypothenuse, } \frac{y}{a} + \frac{x}{b} = 1;$$

therefore, substituting for y

$$\int x^2 y dx = \int a\left(1 - \frac{x}{b}\right) x^2 dx = a\left(\frac{1}{3} - \frac{1}{4} - \frac{x}{b}\right) + C;$$

$$\therefore \int_0^b x^2 y dx = \frac{1}{12} ab^3.$$

$$\text{Similarly, } \int_0^b xy dy = \frac{1}{6} ab^2,$$

$$\int xy^2 dx = \int a^2\left(1 - 2\frac{x}{b} + \frac{x^2}{b^2}\right) x dx = a^2\left(\frac{1}{2} - \frac{2x}{3b} + \frac{x^2}{4b^2}\right) x^2 + C$$

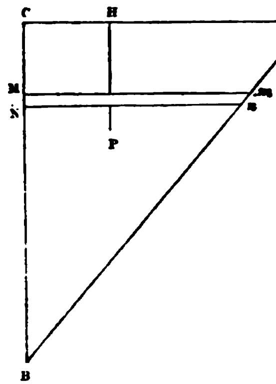
$$= \frac{1}{12} a^2 b^2; \quad \text{(between limits)}$$

$$\therefore x = \frac{1}{2} b, \qquad \bar{y} = \frac{1}{4} a.$$

5. Let $CA = a$, $CB = b$, $CM = x$, $MN = h$,

P the centre of pressure, and $HP = z$.

Now pressure on $Mn = \dfrac{a(b - x)}{b} h.x\rho$ ultimately.



Moment of this pressure about CA

$$= \frac{ah\rho}{b}(b-x)x^2 = \frac{ah\rho}{b}(bx^2 - x^3).$$

For x write h, $2h$, $3h$, ... nh successively; then Sum of all such moments

$$= \frac{ah\rho}{b}[b\{h^2 + (2h)^2 + (3h)^2 + \ldots + (nh)^2\} - \{h^3 + (2h)^3 + (3h^3) + \ldots + (nh)^3\}]$$

$$= \frac{ah\rho}{b}\left\{bh^2 . \frac{n(n+1)(2n+1)}{6} - h^3 . \frac{n^2(n+1)^2}{4}\right\}$$

$$= a\rho\left\{\frac{(nh)^3}{3}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n}\right) - \frac{(nh)^4}{b} . \frac{1}{4}\left(1+\frac{1}{n}\right)^2\right\}$$

$$= ab^3\rho\left(\frac{1}{3}-\frac{1}{4}\right), \text{ when } n \text{ is infinitely great and } nh = b,$$

$$= \frac{ab^3\rho}{12}.$$

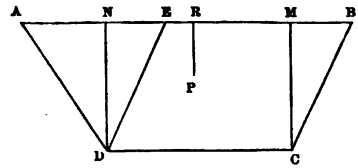
Again, pressure on $ABC = \frac{1}{2}ab . \frac{1}{3}b\rho$,

moment of this pressure about $CA = \frac{ab^2}{6}\rho . z$;

$$\therefore \frac{ab^2}{6}\rho . z = \frac{ab^3}{12}\rho;$$

$$\therefore z = \frac{1}{2}b.$$

In like manner we may find $CH = \frac{1}{4}a$.



6. Let $ABCD$ be the trapezoid, draw DE parallel to CB and CM, DN perpendiculars on AB.

Let $AB = a$, $DC = b$,

$DN = CM = h$,

P the centre of pressure of trapezoid.

Then pressure on $ABCD$ = press. on $NDCM$ + press. on Δ's ADN, BCM,

pressure on $NDCM = bh \times \frac{h}{2}\rho$, which may be considered to act at its centre of pressure, (Vide Ex. 3);

$\therefore$ moment of this pressure about $AB = \frac{1}{2} bh^2\rho \times \frac{2}{3} h = \frac{1}{3} bh^3\rho$,

Δ's ADN, BCM = area of ADE.

moment of press. on ADE about $AB = (a-b)\frac{h}{2} \times \frac{h}{3}\rho \times \frac{h}{2}$, (Vide Ex. 3_5),

$$\text{press. on trapezoid} = (a+b)\frac{h}{2} \times \left(\frac{a+2b}{a+b}\right)\frac{h}{3}\rho;$$

$$\therefore\ RP \times (a+b)\frac{h}{2}\left(\frac{a+2b}{a+b}\right)\frac{h}{3}\rho = \frac{1}{3} bh^3\rho + (a-b)\frac{h^3}{12}\rho = \frac{h^3}{12}(a+3b)\rho;$$

$$\therefore\ RP = \left(\frac{a+3b}{a+2b}\right)\frac{h}{2}.$$

7. The intersection of the plane of the circle with the surface of the fluid being taken as axis of y, and the diameter perpendicular to the surface as axis of x, the formula to be used is

$$\bar{x} = \frac{\int_0^{2r} x^2 y dx}{\int_0^{2r} xy dx},$$

and since $y^2 = (2rx - x^2)$;

$$\therefore \bar{x} = \frac{\int_0^{r} x^2 (2rx - x^2)^{\frac{1}{2}}\, dx}{\int_0^{r} x\, (2rx - x^2)^{\frac{1}{2}}\, dx},$$

$$\text{(between limits)} = \frac{\frac{5}{8} r^4\pi}{\frac{1}{2} r^3\pi} = \frac{5}{4} r.$$

9. Let the centre of the circle be taken as origin of co-ordinates; then, since

$$y = (r^2 - x^2)^{\frac{1}{2}},$$

$$\bar{x} = \frac{\int_0^r x^2 (r^2 - x^2)^{\frac{1}{2}}\, dx}{\int_0^r (r^2 - x^2)^{\frac{1}{2}}\, dx}, \quad \bar{y} = \frac{1}{2} \cdot \frac{\int_0^r x\, (r^2 - x^2)\, dx}{\int_0^r x (r^2 - x^2)^{\frac{1}{2}}\, dx}.$$

$$\text{Assume } x = r \sin\theta,$$

$$\text{then } dx = r \cos\theta\, d\theta, \quad (r^2 - x^2)^{\frac{1}{2}} = r\cos\theta;$$

$$\therefore \int_0^r x^2 (r^2 - x^2)^{\frac{1}{2}}\, dx = r^4 \int_0^{\frac{1}{2}\pi} \sin^2\theta \cos^2\theta\, d\theta.$$

$$\text{Now } \int \sin^2\theta \cos^2\theta\, d\theta = \frac{1}{4}\int \sin^2 2\theta\, d\theta = \frac{1}{8}\int (1 - \cos 4\theta)\, d\theta$$

$$= \frac{1}{8}\left(\theta - \frac{1}{4}\sin 4\theta\right) + C;$$

$$\therefore r^4 \int_0^{\frac{1}{2}\pi} \sin^2\theta \cos^2\theta\, d\theta = \frac{1}{16} r^4\pi,$$

$$\text{and } \int x\, (r^2 - x^2)^{\frac{1}{2}}\, dx = -\frac{1}{3}(r^2 - x^2)^{\frac{3}{2}} + C,$$

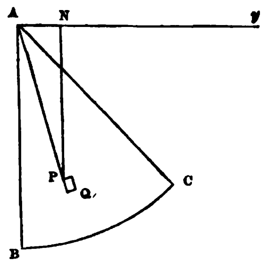
$$\text{between limits} = \frac{1}{3} r^3,$$

$$\text{also } \int_0^r x\, (r^2 - x^2)\, dx = \frac{1}{4} r^4;$$

$$\therefore\ \bar{x} = \frac{\frac{1}{16} r^4 \pi}{\frac{1}{3} r^3} = \frac{3}{16} \pi r,$$

$$\bar{y} = \frac{1}{2} \cdot \frac{\frac{1}{4} r^4}{\frac{1}{3} r^3} = \frac{3}{8} r.$$

10. Let ABC be the sector, the radius BA being perpendicular to the surface of the fluid.



Let $AB = r$, $BAC = \alpha$, $AP = r_1$, any radius-vector inclined at an angle θ to AB, and let PQ be the element described by dr_1, the radius vector $r_1 + dr_1$ having revolved through an angle $d\theta$.

Then $PQ = dr_1 \times d\theta$,

and $p = \rho \,.\, PN = \rho r_1 \cos\theta$;

$$\therefore \text{ pressure on } PQ = \rho r_1^2 dr_1 \cos\theta \, d\theta.$$

Taking the moments about AB and Ay, we have if $\bar{x}$, $\bar{y}$ be the co-ordinates of the centre of pressure, and since

$$NP = r_1 \cos\theta, \quad AN = r_1 \sin\theta,$$

$$\bar{x} \int_0^r \int_0^\alpha r_1^2 \cos\theta \, dr_1 \, d\theta = \int_0^r \int_0^\alpha r_1^3 \cos^2\theta \, dr_1 \, d\theta,$$

$$\bar{y} \int_0^r \int_0^\alpha r_1^2 \cos\theta \, dr_1 \, d\theta = \int_0^r \int_0^\alpha r_1^3 \cos\theta \sin\theta \, dr_1 \, d\theta.$$

$$\text{Now } \iint r_1^3 \cos^2\theta \, dr_1 \, d\theta = \int \frac{1}{4} r_1^4 \cos^2\theta \, d\theta + C;$$

$$\therefore \iint_0^r r_1^3 \cos^2\theta \, dr_1 \, d\theta = \frac{r^4}{4} \int \frac{1}{2} \{1 + \cos 2\theta\} \, d\theta$$

$$= \frac{r^4}{8} \left\{\theta + \frac{1}{2} \sin 2\theta\right\} + C;$$

$$\therefore \int_0^r \int_0^\alpha r_1^3 \cos^2\theta \, dr_1 \, d\theta = \frac{r^4}{8} \left\{\alpha + \frac{1}{2} \sin 2\alpha\right\}.$$

Similarly, $\int_0^r \int_0^\alpha r_1^2 \cos\theta \, dr_1 \, d\theta = \frac{r^3}{3} \sin\alpha.$

Also $\int_0^r \int_0^\alpha r_1^3 \cos\theta \sin\theta \, dr_1 \, d\theta = \int \frac{r_1^4}{4} \cdot \frac{1}{2} \sin 2\theta \, d\theta + C$

$$= \frac{r^4}{8}\{C + \frac{1}{2}\cos 2\theta\} = \frac{r^4}{16}(-\cos 2\alpha + 1) = \frac{r^4}{8}\sin^2\alpha;$$

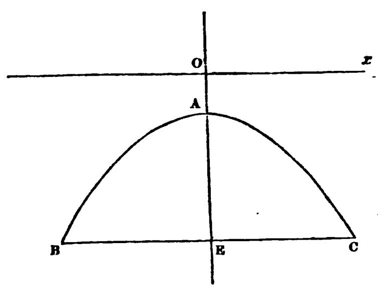
$$\therefore \; \bar{x} = \frac{\frac{r^4}{8}(\alpha + \frac{1}{2}\sin 2\alpha)}{\frac{r^3}{3}\sin\alpha} = \frac{3}{8} r \left\{\frac{\alpha}{\sin\alpha} + \cos\alpha\right\},$$

$$\bar{y} = \frac{\frac{1}{8} r^4 \sin^2\alpha}{\frac{1}{3} r^3 \sin\alpha} = \frac{3}{8} r \sin\alpha.$$

11. The equation to the parabola, referred to the directrix and its axis, as axes of co-ordinates is

$$y^2 = 4a(x - a) \text{......................... (1)},$$

where $4a$ is the latus rectum of the parabola. Let k be the length of the axis of the parabola.



The immersed area being symmetrical with respect to the axis of x, the formula to be used is

$$\bar{x} = \frac{\iint x^2 \, dx \, dy}{\iint x \, dx \, dy}.$$

Integrating with respect to y we have

$$\bar{x} = \frac{\int x^2 \, dx \, . \, y}{\int x \, dx \, . \, y}.$$

Therefore from (1),

$$\bar{x} = \frac{\int_a^{k+a} 2a^{\frac{1}{2}} x^2 (x-a)^{\frac{1}{2}} dx}{\int_a^k 2a^{\frac{1}{2}} x (x-a)^{\frac{1}{2}} dx} = \frac{\int_a^{k+a} x^2 (x-a)^{\frac{1}{2}} dx}{\int_a^k x (x-a)^{\frac{1}{2}} dx}$$

$$= \frac{\frac{2}{105}(x-a)^{\frac{3}{2}}\{15x^2 + 12ax + 8a^2\}}{\frac{2}{15}(x-a)^{\frac{3}{2}}(3x+2a)}$$

$$= \frac{1}{7} \cdot \frac{15x^2 + 12ax + 8a^2}{3x + 2a},$$

and between the limits $x = a + k$, $x = a$,

$$\bar{x} = \frac{3k}{7}\left(\frac{5k + 7a}{3k + 5a}\right).$$

13. From Ex. 1 it appears that the depth of the Centre of Pressure of any rectangle is equal to two-thirds of its height. Hence if we conceive the cylindrical tub to be composed of a great number of similar staves, each of which differs insensibly from rectangular ones, they present a plane surface to the fluid, and the tub when full of fluid will be kept together by a single hoop passing through the centre of pressure of all the staves. Hence the hoop must be at a distance equal to two-thirds the height of the tub from the top.

15. Let r be the radius of the sphere; and the centre of the sphere being taken as origin of co-ordinates, let x, y, z be the co-ordinates of any point P on the surface of the sphere, and the plane yz horizontal.

The pressure on an element at P, resolved in a direction perpendicular to the plane xz is equal to the weight of the column $(x+r)\,dx\,dy$. Similarly the pressure resolved perpendicular to yz = the weight of the column $(x+r)\,dy\,dz$.

Let $\bar{x}$ and $\bar{y}$ be the distances of the resultants of these parallel forces from the planes zy, and xz respectively, then

$$\bar{x} = \frac{\iint (x+r)\,x\,dx\,dz}{\iint (x+r)\,dx\,dz}, \quad \bar{y} = \frac{\iint (x+r)\,y\,dy\,dz}{\iint (x+r)\,dy\,dz}.$$

Now $\iint (x^2 + rx)\,dx\,dz = \int \left(\frac{1}{3}x^3 + \frac{1}{2}rx^2 + C\right) dz$

$$= \int \frac{2}{3}(r^2 - z^2)^{\frac{3}{2}}\,dz,$$

[between the limits $x = -(r^2 - z^2)^{\frac{1}{2}}$ and $x = +(r^2 - z^2)^{\frac{1}{2}}$],

$$= \frac{2}{3}\left\{\frac{z(r^2 - z^2)^{\frac{3}{2}}}{4} + \frac{3r^2}{4}\left(\frac{z}{2}(r^2 - z^2)^{\frac{1}{2}} + \frac{r^2}{2}\sin^{-1}\frac{z}{r}\right)\right\} + C,$$

$$= \frac{1}{4}\pi r^4;\text{ between the limits } z = -r,\ z = +r;$$

$$\iint (x + r)\,dx\,dz = \int \left(\frac{1}{2}x^2 + rx + C\right) dz,$$

$$= \int 2r(r^2 - z^2)^{\frac{1}{2}}\,dz,\text{ limits as before,}$$

$$= 2r\left\{\frac{z(r^2 - z^2)^{\frac{1}{2}}}{2} + \frac{r^2}{2}\sin^{-1}\frac{z}{r}\right\} + C,$$

$$= \pi r^3,\text{ from } z = -r,\text{ to } z = +r.$$

Hence $\bar{x} = \dfrac{\pi r^4}{4\pi r^3} = \dfrac{1}{4}r$..........................(1),

Again,

$$\iint (x + r)\,y\,dy\,dz = \iint \{ry + y(r^2 - y^2 - z^2)^{\frac{1}{2}}\}\,dy\,dz$$

$$= \int \left\{\frac{1}{2}ry^2 - \frac{1}{3}(r^2 - y^2 - z^2)^{\frac{3}{2}} + C\right\} dz$$

$$= \int \left\{\frac{r}{2}(r^2 - z^2) + \frac{1}{3}(r^2 - z^2)^{\frac{3}{2}}\right\} dz$$

limits $y = 0$ to $y = (r^2 - z^2)^{\frac{1}{2}}$,

$$= \frac{1}{2}r\left(r^2 z - \frac{z^3}{3}\right) + \frac{1}{3}\left[\frac{z(r^2 - z^2)^{\frac{3}{2}}}{4} + \frac{3r^2}{4}\left\{\frac{z}{2}(r^2 - z^2)^{\frac{1}{2}} + \frac{r^2}{2}\sin^{-1}\frac{z}{r}\right\}\right] + C$$

$$= \frac{2r^4}{3} + \frac{\pi r^4}{8}\text{ from } z = -r \text{ to } z = +r.$$

Also $\iint(x+r)\,dy\,dz = \iint\{r+(r^2-y^2-z^2)^{\frac{1}{2}}\}\,dy\,dz$

$$=\int\left\{r\,(r^2-z^2)^{\frac{1}{2}}+\frac{\pi}{2}\cdot\frac{r^2-z^2}{2}\right\}dz,$$

limits as before,

$$=\frac{5\pi r^3}{6} \text{ from } z=-r \text{ to } z=+r.$$

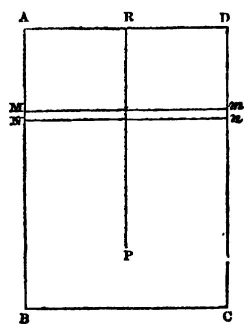
If x be taken negatively, we obtain by repeating the integrations,

$$\iint(r-x)\,dy\,dz=\frac{\pi r^3}{6};$$

$\therefore$ the resultant pressure on the plane of $yz=\dfrac{5\pi r^3}{6}-\dfrac{\pi r^3}{6}=\dfrac{2}{3}\pi r^3$;

$$\therefore\ \bar{y}=\frac{\dfrac{2r^4}{3}+\dfrac{\pi r^4}{8}}{\dfrac{2}{3}\pi r^3}=r\left(\frac{1}{\pi}+\frac{3}{16}\right).$$

$$\text{The resultant}=\left\{\left(\frac{2}{3}\pi r^3\right)^2+(\pi r^3)^2\right\}^{\frac{1}{2}}=\frac{\sqrt{13}\times\pi r^3}{3}.$$



17. Let $AD=a$, $AB=b$;

$AM=x$, $MN=\delta x$,

p the pressure at the depth x.

Then $\rho\propto x^2$, and $=cx^2$ suppose,

$$p=\int\rho dx=c\int_0^x x^2dx=\frac{1}{3}cx^3;$$

$\therefore$ pressure on $Mn=a.\delta x\times\dfrac{1}{3}cx^3$,

moment of this pressure about $AD=a.\delta x\times\dfrac{1}{3}cx^4$;

$$\therefore \text{ sum of all such moments}=\frac{1}{3}ac\int_0^b x^4dx$$

$$=\frac{1}{3}ac\times\frac{1}{5}b^5.$$

$$\text{Again, pressure on } AC=\int_0^b adx\times\frac{1}{3}cx^3$$

$$=\frac{1}{3}ac\times\frac{1}{4}b^4.$$

If now P be the centre of pressure, we have

$$RP\left(\frac{1}{3}ac \times \frac{1}{4}b^4\right) = \frac{1}{3}ac \times \frac{1}{5}b^5;$$

$$\therefore\ RP = \frac{4}{5}b.$$

19. Let the tangent at the vertex and the axis of the parabola be taken as co-ordinate axes. Let Pp be any elemental area of which the sides are dx, dy, and whose depth is x; and let p be the unit of pressure of the fluid on Pp; then, c denoting a constant quantity,

$$p = \int_0^x \rho dx = c\int_0^x x dx = \frac{1}{2}cx^2,$$

and, $dx\,dy$ being the area of Pp,

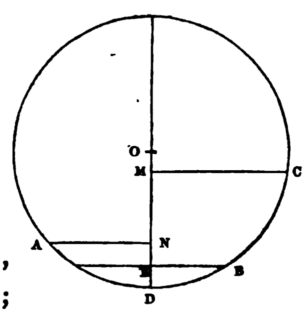
$$\bar{x} = \frac{\iint x^2\, x dx\, dy}{\iint x^2\, dx\, dy},$$

and, integrating with respect to y, since $y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$,

$$\bar{x} = \frac{\int_0^h x^3 dx\, y}{\int_0^h x^2 dx\, y} = \frac{\int_0^h 2a^{\frac{1}{2}}x^{\frac{7}{2}}\,dx}{\int_0^h 2a^{\frac{1}{2}}x^{\frac{5}{2}}\,dx}$$

$$= \frac{\frac{2}{9}h^{\frac{9}{2}}}{\frac{2}{7}h^{\frac{7}{2}}} = \frac{7}{9}h.$$

Ex. 4.



1. Let AB be the heavier fluid, and CB the lighter.

Suppose the radius $= 1$.

Let $\angle AOB = \angle BOC = \alpha$.

By hypothesis $nEN = EM$.

Let $BD = \theta$, then

$$EN = OE - ON = \cos\theta - \cos(\alpha - \theta),$$

$$EM = OE - OM = \cos\theta - \cos(\alpha + \theta);$$

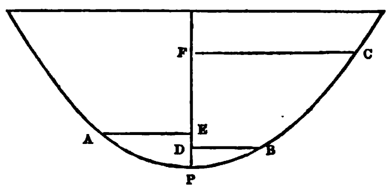
$$\therefore\ n\{\cos\theta - \cos(\alpha - \theta)\} = \cos\theta - \cos(\alpha + \theta);$$

$$\therefore\ n\{1 - \cos\alpha - \sin\alpha\tan\theta\} = 1 - \cos\alpha + \sin\alpha\tan\theta,$$

$$\text{or}\ (n+1)\sin\alpha\tan\theta = (n-1)(1-\cos\alpha);$$

$$\therefore\ \tan\theta = \frac{n-1}{n+1}\cdot\frac{1-\cos\alpha}{\sin\alpha} = \frac{n-1}{n+1}\tan\frac{1}{2}\alpha.$$

3. Let AB be the heavier fluid, and CB the lighter; also let $PE = x_1$, $PE = x_2$, $PD = z$,



then the length of arc AB = length of arc $BC = (2ra)^{\frac{1}{2}}$;

$$\therefore\ 2\sqrt{2rx_1} + 2\sqrt{2rx_2} = 2\sqrt{2ra},$$

$$\text{or}\ \sqrt{x_1} + \sqrt{x_2} = \sqrt{a} \ \ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

Now $\rho'.DE = \rho'.DF$, or $(x_1 - z).\rho = (x_2 - z)\rho'$;

$$\therefore\ z = \frac{x_1\rho - x_2\rho'}{\rho - \rho'}.$$

Also, since $BC = BA$,

$$2\sqrt{2rx_2} - 2\sqrt{2rz} = 2\sqrt{2rx_1} + 2\sqrt{2rz},$$

$$\text{or}\ \sqrt{x_2} - \sqrt{x_1} = 2\sqrt{z} = 2\sqrt{\frac{x_1\rho - x_2\rho'}{\rho - \rho'}} \ \ldots\ldots\ldots\ldots\ (2).$$

From equations (1) and (2) we obtain

$$x_1 = \frac{a}{16}\left(\frac{3\rho + \rho'}{\rho + \rho'}\right)^2, \quad x_2 = \frac{a}{16}\left(\frac{\rho + 3\rho'}{\rho + \rho'}\right)^2.$$

Specific Gravity.

Ex. 5.

1. Let σ, ρ, α be the specific gravities of the cube, water and air respectively, v the volume of cube $= 4^3$ inches.

$$\text{Then } v\sigma - va = 16247 \text{ gr. the weight in air,}$$

$$\text{and } v\sigma - v\rho = 95 \text{ gr. } \ldots\ldots\ldots\ldots\ldots \text{ water;}$$

$$\therefore v\rho - va = 16152;$$

$$\therefore \rho\left(1 - \frac{a}{\rho}\right) = \frac{16152}{64};$$

$$\therefore \rho = \frac{16152}{64} \times \frac{770}{769}, \text{ since } \rho = 770a$$

$$= 252{\cdot}703 \text{ gr., the weight of one cubic inch of water.}$$

2. Let M, M', represent the volumes of the bodies, s, s', their specific gravities, and W, W', their weight; then

$$\begin{array}{l} M : M' = 3^3 : 5^3 \\ s : s' = 3 : 4 \\ \hline \end{array}$$

$$\therefore W : W' = Ms : M's' = 81 : 500.$$

4. $$960 \text{ lb.} = 15360 \text{ oz.};$$

$$\therefore \frac{15360}{1728} = \frac{80}{9} = \text{no. of oz. in a cubic inch of the second metal;}$$

$\therefore$ if s, s' be the specific gravities of the metals,

$$s : s' = 10{\cdot}36 : \frac{80}{9}$$

$$= 1{\cdot}1655 : 1.$$

7. Let M = the volume of the globe, and σ its specific gravity; then its absolute weight $= M\sigma$ and its weight in air is

$$W = M\sigma - Ma = M(\sigma - a),$$

and in water,

$$w = M(\sigma - \rho);$$

$$\therefore \frac{W}{w} = \frac{\sigma - a}{\sigma - \rho},$$

$$\text{whence, } \sigma = \frac{W\rho - wa}{W - w};$$

$$\text{also, } M = \frac{W}{\sigma - a} = \frac{W - w}{\rho - a}.$$

Hence if d be the diameter of the sphere, we have,

$$\frac{\pi}{6} \times d^3 = \frac{W - w}{\rho - a};$$

$$\therefore\ d = \left(\frac{6}{\pi} \cdot \frac{W - w}{\rho - a}\right)^{\frac{1}{3}}.$$

9. Let v, v' be the volumes of gold and diamond,

ρ, ρ' .... sp. grav. ....

Then $v\rho + v'\rho' = 69\frac{1}{2}$ gr., where $\rho = 16{\cdot}5$,

and $v\rho + v'\rho' - (v + v') \times 1 = 64\frac{1}{2}$ gr., and $\rho' = 3{\cdot}5$;

$$\therefore\ v + v' = 5,$$

$$v\rho + v'\rho = 5 \times 16{\cdot}5 = 82{\cdot}5,$$

$$v'(\rho - \rho') = 13;$$

$$\therefore\ v'\rho' = 13 \times \frac{3{\cdot}5}{13} = 3\tfrac{1}{2} \text{ gr.}$$

10. Let v be the volume of gold in the crown,

v' .... silver ....

V .... crown,

ρ, ρ', σ the sp. gr. of the gold, silver, and crown.

$$\text{Now } V\sigma - V \times 1 = \frac{13}{14} V\sigma;$$

$$\therefore\ \sigma = 14.$$

$$\text{Similarly, } \rho = \frac{77}{4} \text{ and } \rho' = \frac{21}{2}.$$

$$\text{But } v\rho + v'\rho' = (v + v')\,\sigma;$$

$$\therefore\ \frac{v}{v'}\rho + \rho' = \left(\frac{v}{v'} + 1\right)\sigma;$$

$$\therefore\ \frac{v\rho}{v'\rho'} = \frac{\sigma - \rho'}{\rho - \sigma} \times \frac{\rho}{\rho'} = \frac{14 - \frac{21}{2}}{\frac{77}{4} - 14} \times \frac{77}{2 \times 21} = \frac{11}{9}.$$

13. Let W = the weight of the hydrometer; then

$W + 60 =$ .... water displaced,

and $W =$ .... spirits displaced,

and the volume is the same in each case, since the instrument is sunk to the same point, and since the specific gravities of two fluids are in the ratio of the weights of equal volumes of the fluids, we have

$$1 : \cdot 866 = W + 60 : W,$$

$$\text{whence } W = \frac{51 \cdot 96}{\cdot 134}$$

$$= 387 \cdot 76 \text{ gr.}$$

16. Let σ be the specific gravity of the mixture.

$$\text{Then } 28\sigma = 21 \times 1 \cdot 84 + 8 \times 1 = 46 \cdot 64;$$

$$\therefore \sigma = 1 \cdot 6657.$$

20. Let V be the volume of the proposed substance, and ρ its specific gravity.

$$\text{Then } \left.\begin{aligned} V\rho - V\lambda &= P \\ V\rho - V\mu &= Q \\ V\rho - V\nu &= R \end{aligned}\right\} \text{ from which equations } V \text{ and } \rho \text{ are to be eliminated;}$$

$$\therefore V(\mu - \lambda) = P - Q,$$

$$\text{and } V(\nu - \mu) = Q - R;$$

$$\therefore \frac{\mu - \lambda}{\nu - \mu} = \frac{P - Q}{Q - R};$$

$$\therefore Q(\mu - \lambda + \nu - \mu) - R(\mu - \lambda) = P(\nu - \mu);$$

$$\therefore P(\nu - \mu) + Q(\lambda - \nu) + R(\mu - \lambda) = 0.$$

21. Let σ be the specific gravity of the mixture, whose

$$\text{volume} = (v + v')\left(1 - \frac{1}{n}\right).$$

$$\text{Then } (v + v')\left(1 - \frac{1}{n}\right)\sigma = v\rho + v'\rho';$$

$$\therefore \sigma = \frac{n}{n-1}\left(\frac{v\rho + v'\rho'}{v + v'}\right).$$

23. Since the surface of a spherical segment is proportional to its height; we have

$$\frac{5}{8} \times 10 \text{ in.} = \text{height of segment immersed in water;}$$

$$\therefore \frac{3}{8} \times 10 \text{ in.} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{naphtha.}$$

Let σ be the sp. gr. of the sphere; then

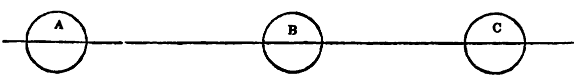
$$\frac{\pi}{6} \times 10^3\sigma = \frac{\pi}{6}\left\{3 \times 10 - 2\left(\frac{5}{8} \times 10\right)\right\}\left(\frac{5}{8} \times 10\right)^2 \times 1$$

$$+ \frac{\pi}{6}\left\{3 \times 10 - 2\left(\frac{3}{8} \times 10\right)\right\}\left(\frac{3}{8} \times 10\right)^2 \times \cdot 708,$$

Vide Coll. of. Ex. p. 97;

$$\therefore \sigma = \left(3 - \frac{10}{8}\right)\left(\frac{5}{8}\right)^2 + \left(3 - \frac{6}{8}\right)\left(\frac{3}{8}\right)^2 \times \cdot 708 = \cdot 9076.$$

25. Let A, B, C be the globes whose sp. grs. are 3, 4, 6 respectively, V the volume of each; x, y, z the distances of their



centres from the fulcrum; then, since they balance one another in vacuo, and in water,

$$3V.x + 4V.y + 6V.z = 0,$$

$$\text{and } (3-1)V.x + (4-1)V.y + (6-1)V.z = 0.$$

$$\text{Hence } 3x + 4y + 6z = 0,$$

$$\text{and } 2x + 3y + 5z = 0;$$

$$\therefore x + y + z = 0;$$

$$\therefore 3x + 2y = 0, \text{ whence } x = -\frac{2}{3}y,$$

$$y + 3z = 0, \ \ldots\ldots \ z = -\frac{1}{3}y.$$

The globes A and C are therefore placed at distances 2 and 1 respectively on one side of the fulcrum, and B at the distance 3 on the other side.

29. Let A be the area of the plane of floatation, W, the weight of the ship, V the volume of water displaced by it, then

$$W = 1{\cdot}026 \times V,$$

and the volume of water displaced on entering the river

$$= V + 2A;$$

$$\therefore\ W = (V + 2A) \times 1,$$

and the volume of water displaced after 12000 lbs. of cargo have been discharged is

$$= V + A;$$

$$\therefore\ W - 12000 = V + A;$$

therefore eliminating A, V, we obtain

$$W\left(1 - \frac{1}{10{\cdot}26}\right) = 12000 \times 2 = 24000,$$

$$\text{whence } W = \frac{24000 \times 1{\cdot}026}{{\cdot}026} = 947076{\cdot}923 \text{ lbs.}$$

$$= 422{\cdot}802 \text{ tons.}$$

32. Let s be the specific gravity of the cylinder, and s', s'' that of the fluids, then

$$s : s'' = \frac{4}{5} : 1, \quad \text{whence } s'' = \frac{5}{4}s,$$

$$\text{and } s : s' = \frac{2}{3} : 1, \quad \text{whence } s' = \frac{3}{2}s;$$

$$\therefore\ \frac{11}{8}s = \text{specific gravity of the mixture.}$$

Let x = the portion of the axis immersed in it, then

$$s : \frac{11}{8}s = x : 1;$$

$$\therefore\ x = \frac{8}{11} \times \text{the axis of the cylinder.}$$

35. Let x = internal radius in inches.

Then weight of iron $= \frac{4\pi}{3}\left\{\left(x+\frac{1}{10}\right)^3 - x^3\right\} \times 7\cdot 2,$

$$\text{............ alcohol} = \frac{4\pi}{3} x^3 \times \cdot 8,$$

$$\text{............ water displaced} = \frac{4\pi}{3}\left(x+\frac{1}{10}\right)^3 \times 1;$$

$$\therefore \left\{\left(x+\frac{1}{10}\right)^3 - x^3\right\} \times 7\cdot 2 + x^3 \times \cdot 8 = \left(x+\frac{1}{10}\right)^3 \times 1;$$

$$\therefore \left(x+\frac{1}{10}\right)^3 \times 6\cdot 2 = x^3 \times 6\cdot 4;$$

$$\therefore 1 + \frac{1}{10x} = \left(\frac{64}{62}\right)^{\frac{1}{3}};$$

whence $x = 9\cdot 4$ in.

36. Let r be the radius of the hemisphere,

.... ρ .... sp. gr. of the fluid,

.... x .... additional weight required.

The solid content of a spherical segment, h being the height, is

$$\frac{\pi}{6}(6r - 2h)h^2 \text{ or } \pi\left(r - \frac{1}{3}h\right)h^2.$$

$$\text{Hence } W = \pi\left\{r - \frac{1}{3}\left(\frac{r}{3}\right)\right\}\left(\frac{r}{3}\right)^2\rho = \frac{8\pi r^3}{81}\rho.$$

$$\text{Also } W + x = \pi\left\{r - \frac{1}{3}\left(\frac{2r}{3}\right)\right\}\left(\frac{2r}{3}\right)^2\rho = \frac{28\pi r^3}{81}\rho;$$

$$\therefore \frac{W+x}{W} = \frac{28}{8}.$$

$$\text{Dividendo } \frac{x}{W} = \frac{20}{8}; \quad \therefore x = 2\cdot 5 \times W.$$

38. Let d be the diameter of the sphere, σ its sp. gr.

$$\text{Then } \frac{\pi}{6} d^3\sigma = \frac{\pi}{6}(3d - 1\cdot 3d)(\cdot 65d)^2 \times 1;$$

$$\therefore \sigma = 1\cdot 7 (\cdot 65)^2 = \cdot 71825.$$

When the air has been admitted, let x be the height of the segment immersed in water; then

$$\frac{\pi}{6}(3d - 2x) x^2 \times 1 + \frac{\pi}{6}\{3d - 2(d-x)\}(d-x)^2 \times \cdot 001225 = \frac{\pi}{6} d^3\sigma;$$

$$\therefore 3dx^2 - 2x^3 + (d^3 - 3dx^2 + 2x^3) \times \cdot 001225 = d^3 \times \cdot 71825;$$

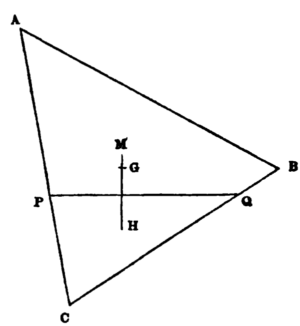
$$\therefore (2x^3 - 3dx^2 + d^3)(1 - \cdot 001225) = d^3 (1 - \cdot 71825);$$

$$\therefore x^3 - 1\cdot 5dx^2 + \cdot 35895d^3 = 0.$$

Equilibrium of Floating Bodies.

Ex. 6.

1. Let ABC and PQ be the sections of the prism and of the plane of floatation; H the centre of gravity of the fluid displaced, G the centre of gravity of the body, and M the metacentre.



Let h be the length of the axis of the prism; then

The moment of inertia of the plane of floatation about an axis through its centre of gravity perpendicular to ABC

$$= \frac{1}{3} \cdot \frac{1}{4} PQ^2 . PQ . h,$$

and the volume of the fluid displaced

$$= \frac{1}{2} h . PC . CQ . \sin C = h \times \text{area of } \Delta PQC;$$

$$\therefore h \times \Delta PQC \times HM = \frac{1}{12} PQ^3 \times h;$$

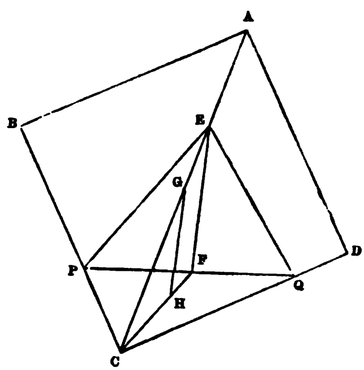
$$\therefore\ HM = \frac{1}{12} PQ^3 \div (\text{area of } \Delta\, PQC)$$

$$= \frac{1}{12}\ (\text{line of floatation})^3 \div \text{area of triangular part im-}$$

mersed.

4. Let $ABCD$ be the square, floating with one angle C immersed in water. Let PQ be the line of floatation; $CP = x$, $CQ = y$. Join AC and bisect it in G.

Bisect PQ in F, join CF, and take $HF = \frac{1}{3} CE$.



Then G and H will be the centre of gravity of $ABCD$, and CPQ respectively. Join GH and draw FE parallel to HG. Then, in order that the lamina may be at rest, G and H must be in the same vertical line; hence EF must be vertical and therefore perpendicular to PQ; therefore the lines joining PE, QE will be equal.

Let a be the length of a side of the square.

Then, the weight of the displaced fluid being equal to that of the square, we have

$$\frac{1}{2} xy = \rho a^2, \text{ or } xy = 2\rho a^2 \ldots\ldots\ldots\ldots\ (1).$$

$$\text{Again, } PE^2 = x^2 + CE^2 - 2x \,.\, CE \cos 45^\circ,$$

$$QE^2 = y^2 + CE^2 - 2y \,.\, CE \cos 45^\circ;$$

$$\therefore \text{ since } PE = QE,$$

$$(x - y)\,(x + y - 2\,CE \cos 45^\circ) = 0$$

$$\text{but } CE : CG = CF : CH;$$

$$\therefore\ CE = \frac{3}{2}\, CG;$$

$$\therefore\ 2CE\cos\frac{\pi}{4} = 3CG\cos\frac{\pi}{4} = \frac{3}{2}a,$$

$$\text{hence } (x-y)\left(x+y-\frac{2}{3}a\right) = 0,$$

which equation gives

$$\text{either } x - y = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\text{or } x + y - \frac{3}{2}a = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Taking (1) and (2) we have

$$x^2 = 2\rho a^2 = y^2,$$

which gives one position of equilibrium if ρ be less than $\frac{1}{2}$.

Taking (1) and (3) we have

$$(x-y)^2 = \frac{a^2}{4}(9 - 32\rho),$$

$$\text{or } x - y = \pm\frac{a}{2}(9-32\rho)^{\frac{1}{2}},$$

$$\text{and } x + y = \frac{3}{2}a;$$

$$\therefore\ x = \frac{a}{4}\{3 \pm (9-32\rho)^{\frac{1}{2}}\},$$

$$y = \frac{a}{4}\{3 \mp (9-32\rho)^{\frac{1}{2}}\}.$$

Thus there will be two other positions of equilibrium if x and y be both possible and positive quantities less than a; that is, if

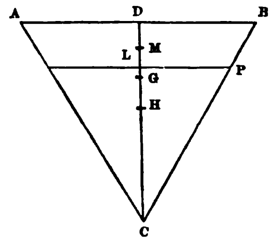
$$\rho < \frac{9}{32}, \text{ and } (9-32\rho)^{\frac{1}{2}} < 1,$$

$$\text{that is, if } \rho \text{ lie between } \frac{8}{32} \text{ and } \frac{9}{32}.$$

There are therefore three positions of equilibrium when one angle is immersed, and therefore twelve for the whole square.

(2) The second case may be deduced from the first by writing $(1-\rho)$ for ρ.

7. The nature of the equilibrium evidently depends on the position of M the metacentre.



Let $CL=x$, $PL=y$ be the height and radius of base of the cone immersed; and $CD=a$, $AD=b$, of the whole cone. Let H be the centre of gravity of the fluid displaced, and G that of the whole cone; then the moment of inertia of the plane of floatation (PQ) round a horizontal axis throughout its centre of gravity L, is

$$=\frac{1}{4}\pi y^4,$$

$$\text{also, } GH=\frac{3}{4}(a-x),$$

and the volume of the cone immersed is

$$=\frac{1}{3}\pi y^2 x,$$

$$\text{hence } HM=\frac{3}{4}\frac{y^2}{x};$$

$$\therefore\ GM=\frac{3}{4}\left\{\frac{y^2}{x}-(a-x)\right\}$$

$$=\frac{3}{4}\left\{\frac{y^2}{x}-a\left(1-\frac{x}{a}\right)\right\}.$$

But the part immersed bears a constant ratio to the whole cone; and the cones being similar solids are to each other as the cubes of their heights or of the radii of their bases, hence

$$x^3=\frac{1}{8}a^3 \text{ and } y^3=\frac{1}{8}b^3;$$

$$\therefore\ \frac{y^2}{x}=\frac{1}{2}\cdot\frac{b^2}{a};$$

$$\therefore\ GM=\frac{3}{4}b\left\{\frac{1}{2}\cdot\frac{b}{a}-\left(1-\frac{1}{2}\right)\right\}=\frac{3}{8}b\left\{\frac{b}{a}-1\right\}.$$

Now since the cone is right-angled at the vertex we have

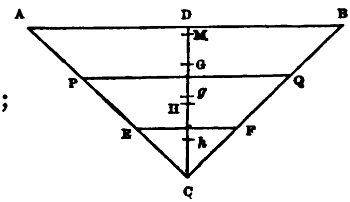
$$a = b,$$

and therefore

$$GM = 0,$$

or, M coincides with G, and the equilibrium is neutral.

9. Let ABC be the cone, G its centre of gravity; g the centre of gravity of the portion in the lighter fluid, h that of the part immersed in the heavier, PQ the plane of floatation.



Let a = axis of the cone equal rad. of base, then

$$Ch = \frac{1}{4}a, \quad Cg = \frac{15}{28}a, \quad CG = \frac{3}{4}a;$$

$$\therefore\ gh = \frac{2}{7}a.$$

Let ρ = density of the lighter fluid, H the centre of gravity of the fluid displaced, and let $gH = x$, then we have

$$x \times \frac{7}{81}\pi a^3 \rho = \frac{1}{81}\pi a^3 \times \frac{3}{2}\rho\left(\frac{2}{7}a - x\right),$$

$$\text{whence } x = gH = \frac{6a}{119};$$

$$\therefore\ CH = \frac{15}{28}a - \frac{6}{119}a = \frac{1617}{3332}a,$$

$$\text{and therefore } HG = \frac{882}{3332}a = \frac{441}{1666}a.$$

Now the mom. inert. of the plane of floatation is

$$= \frac{4}{81}\pi a^4,$$

and the volume of the fluid displaced $= \dfrac{8}{81}\pi a^3$;

$$\therefore\ GM = \frac{\frac{4}{81}\pi a^4}{\frac{8}{81}\pi a^3} - \frac{441}{1666}a = \frac{1}{2}a - \frac{441}{1666}a$$

$$= \frac{196}{833}a, \text{ a positive quantity,}$$

or M is above G, therefore *the equilibrium is stable.*

11. Referring to the figure in Example 7, we have, if I be the mom. inert. of the plane of floatation, and V the volume of the fluid displaced,

$$I=\frac{1}{4}\pi y^4, \quad V=\frac{1}{3}\pi y^2x^2;$$

$$\therefore\ HM=\frac{I}{V}=\frac{3}{4}\cdot\frac{y^2}{x},$$

and by the conditions of the question

$$\frac{3}{4}\frac{y^2}{x}=\frac{1}{4}x;$$

$$\therefore\ y^2=\frac{1}{3}x^2;$$

$$\text{whence } \frac{y}{x}=\frac{1}{\sqrt{3}}=\tan 30^\circ;$$

therefore the vertical angle of the cone is 60°.

15. Let x, y be the height and radius of the base of the paraboloid immersed, and a, b, of the whole paraboloid, then volume of the fluid displaced is

$$=\frac{1}{2}\pi y^2x,$$

and the volume of the whole paraboloid is

$$=\frac{1}{2}\pi b^2a.$$

And from the condition of equilibrium we get

$$\frac{1}{2}\pi b^2a\times\rho=\frac{1}{2}\pi y^2x\times 3\rho;$$

$$\text{whence } y^2x=\frac{1}{3}b^2a \dots\dots\dots\dots (1),$$

$$\text{but } y^2=lx \text{ and } b^2=la;$$

therefore, substituting in (1), we get

$$x=\left(\frac{1}{3}\right)^{\frac{1}{2}}\times a, \text{ the depth required.}$$

18. Using the same notation as in the preceding example, we have
moment of inertia of plane of floatation

$$=\frac{1}{4}\pi y^4,$$

volume of the fluid displaced

$$=\frac{1}{2}\pi y^2 x;$$

$$\therefore\ HM=\frac{\frac{1}{4}\pi y^4}{\frac{1}{2}\pi y^2 x}=\frac{1}{2}\cdot\frac{y^2}{x},$$

$$\text{and } HG=\frac{2}{3}(a-x);$$

$$\therefore\ GM=\frac{1}{2}\cdot\frac{y^2}{x}-\frac{2}{3}(a-x).$$

Let $l=$ the latus rectum of the generating parabola, then, since $x=\frac{1}{2}a$,

$$\therefore\ GM=\frac{1}{2}l-\frac{2}{3}\left(a-\frac{1}{2}a\right),$$

and for the position of indifferent equilibrium we have

$$\frac{1}{2}l-\frac{1}{3}a=0;$$

$$\text{whence } \frac{a}{l}=\frac{3}{2}.$$

ELASTIC FLUIDS.

Ex. 7.

1. Let ρ be the density of the air at any point P above the earth's surface, c the compressive force of the air at that point, or the weight W of the superincumbent air, $h=$ the altitude of an homogeneous atmosphere at P, upon supposition that the density is uniform, then

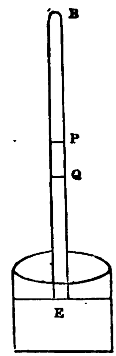
$$W\propto\rho\times h;$$

$$\therefore\ h \propto \frac{W}{\rho} \propto \frac{c}{\rho},$$

but $c \propto \rho$;

hence h is constant.

2. Let BP be the space occupied by the air in the tube at first; and when the cylinder is immersed in the water, let the water stand at Q; then the air which before occupied the space BP, now fills the space BQ. And since the elasticity of the air is proportional to the density, and the density is inversely as the space occupied by the same quantity, we have



elasticity of air in BP : elasticity in $BQ = BQ : BP$.

Now $BE = 20$ ft., $BP = 10$ ft., and the elasticity of the air when it occupies the space BP would support a column of water 33 ft.; also the elasticity of the air in BQ + the weight of the water in QE, are balanced by the pressure of the atmosphere, and therefore would support a column of water 33 ft.; hence, if $QE = x$, the elasticity of the air in BQ would support a column of water $= 33 - x$; hence

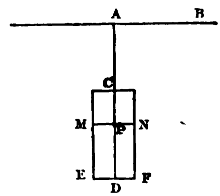
$$33 : 33 - x = 20 - x : 10,$$

$$\text{or } x^2 - 53x = -330,$$

$$\text{whence } x = 7{\cdot}206,$$

the less root alone satisfying the conditions of the problem.

6. Let AB represent the surface of the water; CD the cylindrical tube; DP the height to which the water rises in the tube. Let V be the content of the whole tube, V' the content of the portion above MPN. When the air in the tube occupies the space CP, the pressure of the water is proportional to the depth $AP = x + 1$, and the pressure of the atmosphere is proportional to 32 ft.; therefore the whole pressure upon the air is equal to the weight of a column of water whose height is



$$32 + x + 1 = 33 + x.$$

But the elastic force of the air is as its density, or inversely

as the space occupied when the quantity remains the same; therefore

$$4 : 1 :: V : V' :: 33 + x : 32;$$

$$\therefore\ 33 + x = 128,$$

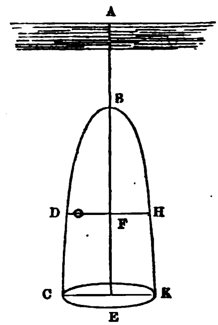
$$\text{and } x = AC = 95 \text{ ft.}$$

9. Let $AB = h$,

$EF = k$,

$EB = a$,

a = density of atmosphere,



$$\text{press. at } D = 33 + h + a - k,$$

$$\frac{\text{press. of air in } BDH}{\text{press. of air in } BCK} = \frac{\text{vol. } BCK}{\text{vol. } BDH},$$

$$\text{vol. } BCK = \frac{\pi}{2} . CE^2 . BE = \frac{\pi}{2} . L . BE^2;$$

L being the latus rectum;

$$\text{vol. } BDH = \frac{\pi}{2} . L . BF^2;$$

$$\therefore\ \frac{\text{press. of air in } BDH}{\text{press. of air in } BCK} = \frac{BE^2}{BF^2} = \left(\frac{a}{a-k}\right)^2.$$

$$\text{But } \frac{\text{press. of air in } BCK}{\text{press. of atmosphere}} = \frac{\rho}{\alpha};$$

$$\therefore\ \text{press. of air in } BCK = \frac{\rho}{\alpha} \times 33.$$

$$\text{Hence } 33 + h + a - k = \left(\frac{a}{a-k}\right)^2 \frac{\rho}{\alpha} \times 33;$$

$$\therefore\ \rho = \left(1 + \frac{h + a - k}{33}\right)\left(1 - \frac{k}{a}\right)^2 \alpha.$$

10. Let p represent the unit of pressure at every point of the air before, and p' after, the insertion of the piston.

Then, the pressure on the base of the piston being $\pi \times \frac{9}{4} \times p'$, we have

$$\pi \times \frac{9}{4} \times p' = 5 + \pi \times \frac{9}{4} \times p \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let x be the distance of the piston from the top of the cylinder, when it has ceased to descend; then we shall have

$$p' : p = 24 : 24 - x,$$

$$\text{or } p'(24 - x) = 24p,$$

and therefore from (1) we have

$$x = \frac{24 \times 80}{80 + \pi \times \frac{9}{4} \times p},$$

$$\text{but } p = \frac{13568 \times 30}{1728} 235{\cdot}5;$$

$$\therefore\ x = \frac{1920}{1745{\cdot}048} = 1{\cdot}1 \text{ in. nearly.}$$

12. Here $h = 30{\cdot}151$, $t = 59{\cdot}9$, $T = 59{\cdot}9$, $h' = 26{\cdot}474$, $t' = 49{\cdot}1$, and $T' = 50{\cdot}88$;

$$\therefore\ z = 60345 \left(1 + \frac{109 - 64}{900}\right) \log \frac{30{\cdot}151}{26{\cdot}474\{1 + 9{\cdot}02 \times {\cdot}0001\}}$$

$$= 63362{\cdot}25 \times \log \frac{30{\cdot}151}{26{\cdot}49788}$$

$$= 63362{\cdot}25 \times {\cdot}056091 = 3554{\cdot}052 \text{ feet.}$$

14. Let W be the weight of the balloon with all its appendages; and let x denote the specific gravity, or density, and z' the height of that stratum of the atmosphere at which the balloon will cease to ascend.

Then since the weight of the air displaced by the gas

$$= \frac{4}{3} \pi r^3 \times x,$$

and that of the balloon and its appendages and gas is

$$W + \frac{4}{3} \pi r^3 \times a;$$

therefore, by the question,

$$\frac{4}{3} \pi r^3 x = W + \frac{4}{3} \pi r^3 a,$$

$$\text{whence}\quad x = \frac{3W}{4\pi r^3} + a;$$

therefore, if the temperatures at the height z, and at the surface of the earth be considered the same, we have

$$z = 60345 \log \frac{1}{\frac{3W}{4\pi r^3} + a}$$

$$= 60345 \log \frac{4\pi r^3}{3W + 4\pi r^3 a}.$$

16. Let x = radius of balloon.

$$\text{Then} \quad 2 \times 5280 = \frac{k}{M} \log \frac{H}{h}.$$

$$7 \times 5280 = \frac{k}{M} \log \frac{4}{1};$$

$$\therefore \frac{2}{7} = \frac{\log \frac{H}{h}}{\log 4}; \quad \therefore \log \frac{H}{h} = \frac{2}{7} \log 4 = \log 16^{\frac{1}{7}};$$

$$\therefore \frac{H}{h} = 16^{\frac{1}{7}}.$$

$$\text{But} \quad \frac{\rho_n}{\rho} = \frac{h}{H}; \qquad \therefore \rho_n = \frac{1 \cdot 2}{16^{\frac{1}{7}}} \text{ oz.}$$

Now weight of gas $= \frac{4\pi}{3} x^3 \times \cdot 12$ oz.,

.... appendages $= 800 \times 16$ oz.,

.... air displaced $= \frac{4\pi}{3} x^3 \times \rho_n$;

$$\therefore \frac{4\pi}{3} x^3 \times \frac{1 \cdot 2}{16^{\frac{1}{7}}} = \frac{4\pi}{3} x^3 \times \cdot 12 + 800 \times 16;$$

$$\therefore x^3 . \pi \left(\frac{1 \cdot 6}{16^{\frac{1}{7}}} - \cdot 16 \right) = 800 \times 16;$$

$$\therefore x = \left\{ \frac{800 \times 16^{\frac{8}{7}}}{\pi \left(1 \cdot 6 - \frac{16^{\frac{8}{7}}}{100} \right)} \right\}^{\frac{1}{3}}$$

$$= 16 \cdot 44 \text{ ft.}$$

17. $$z = \frac{k(1 + \alpha\tau)}{Mg} \log \frac{H}{h}.$$

Now, weight of copper $= \frac{4\pi}{3}\left\{\left(50 + \frac{1}{1200}\right)^3 - 50^3\right\} 8788$ oz.,

.... gas $= \frac{4\pi}{3} \times 50^3 \times \cdot 1225$ oz.,

.... air displaced $= \frac{4\pi}{3}\left(50 + \frac{1}{1200}\right)^3 \rho_n$,

.... car, &c. $= 5000 \times 16$ oz.

$$\therefore \left(50 + \frac{1}{1200}\right)^3 \rho_n = \left\{\left(50 + \frac{1}{1200}\right)^3 - 50^3\right\} 8788 + 50^3 \times \cdot 1225 + \frac{60000}{\pi},$$

whence the value of ρ_n is found.

$$\text{Now } \frac{H}{h} = \frac{\rho}{\rho_n} = \frac{1\cdot 225}{\rho_n};$$

$$\therefore z = 60345 \log \frac{1\cdot 225}{\rho_n}$$

$$= 14123 \text{ ft. or } 14623 \text{ ft.},$$

according as small terms or none are omitted in computing the value of ρ_n.

INSTRUMENTS AND MACHINES.

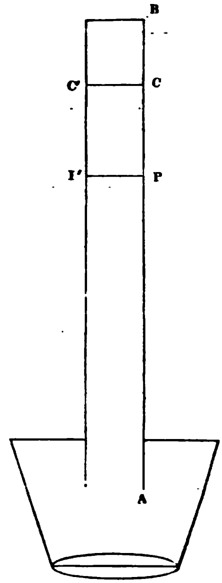
Ex. 8.

1. WHEN the fluid in the tube falls 1 inch, that in the cistern rises

$$\frac{\pi \times 1^2}{\pi \times 3^2} \times 1 = \frac{1}{9} \text{ in.};$$

therefore the true variation

$$= 1 + \frac{1}{9} = \frac{10}{9} \text{ in.}$$



3. $AB = 33$ in., $AC = 29$ in., $AP = x$,

pressure of air in $BC = 30 - 29 = 1$ in.

Again,

$$\frac{\text{pressure of air in } BP'}{\text{....................} BC'} = \frac{BC}{BP} = \frac{4}{33 - x};$$

$$\therefore \text{ pressure in } BP' = \frac{4 \times 1}{33 - x}.$$

$$\text{Hence } \frac{4 \times 1}{33 - x} + x = 25;$$

$$\therefore (33 - x)(25 - x) = 4,$$

$$x^2 - 58x + 29^2 = 841 - 825 + 4 = 20,$$

$$x = 29 \pm 4{\cdot}47213$$

$$= 24{\cdot}52787 \text{ in.}$$

5. Let x, y be the length of tube, which the air left in the barometers, of the mean density of the atmosphere, would occupy; h, k, the heights at which the perfect instrument would stand on the two days; then we have the following equations:

$$(1) \quad (h - a)(l - a) = hx; \qquad (3) \quad (h - a')(l - a') = hy;$$

$$(2) \quad (k - b)(l - b) = kx; \qquad (4) \quad (k - b')(l - b') = ky;$$

whence, eliminating h and k, we get

$$x = \left\{\frac{a'(l + a - a')}{a} - \frac{b'(l + b - b')}{b}\right\} \div \left\{\frac{a'(l - a')}{a(l - a)} - \frac{b'(l - b')}{b(l - b)}\right\},$$

$$y = \left\{\frac{a(l + a' - a)}{a'} - \frac{b(l + b' - b)}{b'}\right\} \div \left\{\frac{a(l - a)}{a'(l - a')} - \frac{b(l - b)}{b'(l - b')}\right\}.$$

7. Generally, $\rho_n = \left(\frac{R}{R + b}\right)^n \rho$;

$$\therefore \rho_3 = \left(\frac{R}{R + b}\right)^3 \rho.$$

Therefore, by the question, we have

$$\left(\frac{R}{R+b}\right)^3 \rho = \frac{1}{4}\rho;$$

$$\therefore \frac{R}{R+b} = \frac{1}{4^{\frac{1}{3}}},$$

whence $R : b = 1 : 4^{\frac{1}{3}} - 1$.

9. Let A be the air-pump with barrel of 1 foot capacity,

B .... 2 feet ....

ρ_5 be the density of air in A after 5 turns,

ρ_3' .... B ...... 3 ......

Then $\rho_5 = \left(\frac{10}{10+1}\right)^5 \rho = \left(\frac{10}{11}\right)^5 \rho$; $\rho_3' = \left(\frac{10}{10+2}\right)^3 \rho = \left(\frac{10}{12}\right)^3 \rho$,

(P) quantity of air exhausted from A

= quantity of air at first − quantity after 5 turns,

$= 10\rho - 10\rho_5$.

Similarly, (Q) the quantity exhausted from B

$= 10\rho - 10\rho_3'$.

Hence,

$$\frac{P}{Q} = \frac{10\rho - 10\rho_5}{10\rho - 10\rho_3'} = \frac{1 - \left(\frac{10}{11}\right)^5}{1 - \left(\frac{10}{12}\right)^3} = \frac{12^3}{11^5} \times \frac{11^5 - 10^5}{12^3 - 10^3} = \frac{9}{10} \text{ nearly.}$$

11. Let x be the number of turns; then after x turns the quantity of air in the pump

$$= \left(\frac{R}{R+b}\right)^x p;$$

therefore by the question we have

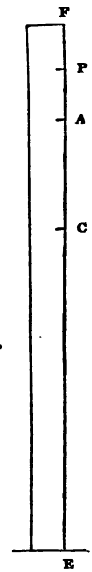
$$\left(\frac{R}{R+b}\right)^x p = p - q;$$

$$\therefore\ x \log\left(\frac{R}{R+b}\right) = \log\left(1 - \frac{q}{p}\right);$$

$$\therefore\ x = \log\left(1 - \frac{q}{p}\right) \div \log\left(\frac{R}{R+b}\right).$$

14. Let $EF = l$, $EA = a$, $EC = c$, $FP = x$;

h the standard altitude, or height of mercurial column in the true barometer.



$$\text{Then } \frac{\text{pressure of air in } FA}{\text{pressure of air in } FP} = \frac{FP}{FA} = \frac{x}{l-a};$$

$$\therefore \text{ pressure of air in } FA = \frac{hx}{l-a};$$

$$\therefore\ \frac{hx}{l-a} + a = \text{pressure at } E = h \ \ldots\ldots\ldots\ldots\ldots (1).$$

If ρ be the density of atmospheric air in the receiver at first,

and ρ_n be the density of this air after n turns,

$$\text{then } \rho_n = \left(\frac{R}{R+b}\right)^n \rho = \left(\frac{m}{m+1}\right)^n \rho = q\rho \text{ suppose.}$$

Now if h measure the pressure of the air of density ρ,
and h' .... ρ_n,

$$\text{we have } \frac{h'}{h} = \frac{\rho_n}{\rho}; \quad \therefore\ h' = qh.$$

$$\text{Also } \frac{\text{pressure of air in } FC}{\text{pressure of air in } FP} = \frac{FP}{FC} = \frac{x}{l-c};$$

$$\therefore\ \frac{hx}{l-c} + c = \text{pressure at } E \text{ under the receiver} = qh \ \ldots\ldots\ldots (2).$$

Equations (1) and (2) give the values of h and x; thus

$$(qh - c)(l - c) = hx = (h - a)(l - a);$$

$$\therefore\ h\,\{q(l-c) - (l-a)\} = c(l-c) - a(l-a);$$

$$\therefore\ h = a\left(\frac{c}{a} - \frac{l-a}{l-c}\right) \div \left\{\left(\frac{m}{m+1}\right)^n - \frac{l-a}{l-c}\right\}, \text{ replacing } q \text{ by } \left(\frac{m}{m+1}\right)^n.$$

$$\text{Also } x = (l-a)\left\{\frac{c}{a} - \left(\frac{m}{m+1}\right)^n\right\} \div \left(\frac{c}{a} - \frac{l-a}{l-c}\right).$$

16. Let R be the capacity of the receiver of the condenser, and also the capacity of the receiver of the air-pump, b the capacity of the barrel of the air-pump, and ρ the density of the common air in the receiver and barrel at first. Then the mass of air in the receiver of the condenser at first is $R\rho$. The mass of air in the receiver of the air-pump after n turns is

$$\left(\frac{R}{R+b}\right)^n R\rho;$$

therefore the mass of air discharged into the receiver of the condenser after n turns is

$$R\rho - \left(\frac{R}{R+b}\right)^n R\rho;$$

and therefore the mass of air in the receiver of the condenser after n turns of the air-pump is

$$R\rho - \left(\frac{R}{R+b}\right)^n R\rho + R\rho = \left\{2 - \left(\frac{R}{R+b}\right)^n\right\} R\rho.$$

Therefore the density of the air in the receiver after n turns is

$$\left\{2 - \left(\frac{R}{R+b}\right)^n\right\} \rho.$$

18. Let ρ be the density of the air in the receiver at first, and consequently of the air in the gauge. After 12 turns the density of the air in the receiver is

$$\frac{30b + 12b}{30b}\rho = \frac{7}{5}\rho.$$

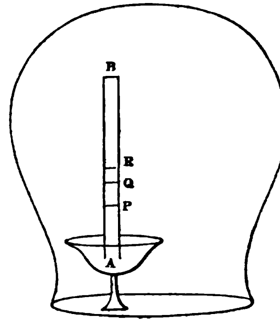
Now the density of the air in the receiver increases in arithmetical progression, and as the density is inversely as the space it occupies, we shall have, if x be the distance of the globule of mercury from the receiver after 12 turns,

$$20 : 20 - x = \frac{7}{5}\rho : \rho;$$

$$\therefore\ 100 = 140 - 7x;$$

$$\therefore\ 7x = 40,$$

$$x = 5\tfrac{5}{7} \text{ inches}.$$



19. Let $AB = 20$ in.,

$AP = x_1$,

$AQ = x_2$,

$AR = x_3, \ldots$

$\rho, \rho_1, \rho_2, \ldots\ldots$ the densities of the air in the receiver at first, and after one, two, &c. descents of the piston.

$$\text{Then } \rho_1 = \frac{R+b}{R}\rho = \frac{11}{10}\rho,$$

$$\rho_2 = \frac{12}{10}\rho, \quad \rho_3 = \frac{13}{10}\rho, \quad \&c.$$

$$\text{Now } \frac{\text{pressure of air in } BP}{\text{pressure of air in } BA} = \frac{BA}{BP} = \frac{20}{20 - x_1};$$

$\therefore$ pressure of air in $BP = \dfrac{20 \times 30}{20 - x_1}$, since the barometer stands at 30 in.;

$$\therefore \text{ press. at } A = \frac{20 \times 30}{20 - x_1} + x_1.$$

$$\text{But the } \frac{\text{pressure of air in the receiver}}{\text{pressure of external air}} = \frac{\rho_1}{\rho} = \frac{11}{10},$$

$$\therefore \text{ pressure at } A = \frac{11}{10} \times 30 = 33.$$

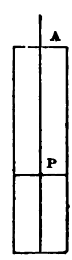
$$\text{Hence } \frac{20 \times 30}{20 - x_1} + x_1 = 33,$$

$$(20 - x_1)(33 - x_1) = 20 \times 30; \quad \therefore x_1 = 1{\cdot}16.$$

Similarly, $(20 - x_2)(36 - x_2) = 20 \times 30; \quad \therefore x_2 = 2{\cdot}2318,$

$(20 - x_3)(39 - x_3) = 20 \times 30; \quad \therefore x_3 = 3{\cdot}2274,$

&c. = &c. &c.



21. In the figure, let P be the position of the piston when the valve in it opens after r descents,

$$AB = a, \quad AP = x_r.$$

When the piston is at A after r descents, the density of the air in AB is expressed by

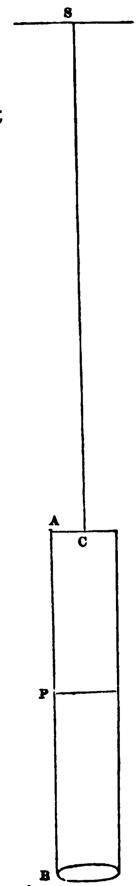
$$\rho_r = \left(\frac{R}{R+b}\right)^r \rho.$$

When the piston has descended to P, the density of the air in PB has increased till it equals the density of the external air, after which the valve in the piston opens;

$$\therefore \frac{\rho_r}{\rho} = \frac{PB}{AB} = \frac{a - x_r}{a};$$

$$\therefore 1 - \frac{x_r}{a} = \left(\frac{R}{R+b}\right)^r,$$

$$x_r = a\left\{1 - \left(\frac{R}{R+b}\right)^r\right\}.$$



22. Let h, h' be the altitudes of mercury at S and P respectively.

$$SC = x.$$

Then $$\frac{\text{pressure of air in } AP}{\text{..................... } AB} = \frac{AB}{AP} = \frac{h'}{h};$$

$$\therefore AP = \frac{ah}{h'}.$$

But pressure at P

$$= h + \frac{x}{\sigma} + \frac{ah}{h\sigma} = h';$$

$$\therefore x = (h' - h)\,\sigma - \frac{ah}{h'};$$

B has therefore descended through

$$x + a = (h' - h)\,\sigma + a\left(1 - \frac{h}{h'}\right)$$

$$= (h' - h)\left(\sigma + \frac{a}{h'}\right).$$

23. Let V be the content of the bell, V' that of the part of the bell which is clear of water, h (= 33 ft.) the height of a

column of water whose pressure is equal to that of the atmosphere; and let x be the depth to which the top of the bell is sunk; (a) the radius of the hemisphere.

Then the pressure at any point of the surface of the water in the bell is the weight of a column $h+x+\frac{a}{2}$, being as the depth below the surface of the water; and the elastic force of the air being inversely as the space occupied, we have

$$h : h+x+\frac{a}{2}=\frac{1}{V} : \frac{1}{V'}=V' : V \ldots\ldots\ldots\ldots\ldots(1).$$

$$\text{Now } V=\frac{2}{3}\pi a^3, \text{ and } V'=\frac{\pi}{6}(6a-a)\frac{a^2}{4}=\frac{5}{24}\pi a^3.$$

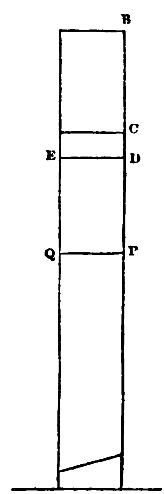
Therefore, substituting in (1), we get

$$33 : 33+\left(x+\frac{a}{2}\right)=\frac{5}{24}\pi a^3 : \frac{2}{3}\pi a^3;$$

$$\therefore 33 : x+\frac{a}{2}=\frac{5}{24} : \frac{11}{24}=5 : 11;$$

$$\therefore x+\frac{a}{2}=\frac{11\times 33}{5}=\frac{363}{5}=72\cdot 6 \text{ ft.}$$

25. Let $AB=20$ ft.,
$AC=17$ ft.,
$AD=16$ ft.



(1) When the range of piston is BD.

Let P be the greatest weight, and $AP=x$,

$$\frac{\text{pressure of air in } DQ}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ BQ}=\frac{BP}{DP}=\frac{20-x}{16-x}.$$

But pressure in $BQ+x=33$;

$$\therefore \text{pressure in } DQ=\frac{(20-x)(33-x)}{16-x}=33,$$

$$20\times 33-53x+x^2=16\times 33-33x,$$

$$x^2-20x+10^2=-4\times 33+100=-32;$$

$$\therefore x=10\pm\sqrt{-32}.$$

Hence, there is in this case no greatest height, and the water flows out.

(2) When the range of piston is BC.

$$\text{Similarly, } \frac{(20-x)(33-x)}{17-x} = 33;$$

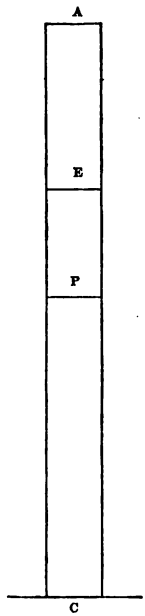
$$\therefore\ x^2 - 53x + 20 \times 33 = 17 \times 33 - 33x;$$

$$\therefore\ x^2 - 20x + 10^2 = 100 - 99 = 1;$$

$$\therefore\ x = 10 \pm 1 = 11 \text{ or } 9.$$

Hence $AP = 11$ ft. or 9 ft., according as the piston is at B or C respectively; the valve at A being open.

26. Let $AC = c$, AE the range of the piston $= x$.



Let $AP = z$, P being the surface of the water in the pump, after any stroke.

Now, pressure of air in AP + weight of water column CP = pressure at C = pressure of atmosphere;

$$\therefore \text{ pressure of air in } AP = a - (c - z).$$

On the piston descending to E,

$$\frac{\text{pressure of air in } EP}{\text{pressure of air in } AP} = \frac{AP}{EP} = \frac{z}{z-x};$$

$$\therefore \text{ pressure of air in } EP = \frac{z(a-c+z)}{z-x}.$$

Now this pressure must be greater than the pressure of the atmosphere, or the valve in the piston at E would not open, and the action of the pump has ceased.

$$\text{Hence } \frac{z(a-c+z)}{z-x} \text{ must be greater than } a;$$

$$\therefore\ az - cz + z^2 > az - ax;$$

$$\therefore\ ax > cz - z^2.$$

But $cz - z^2$ is a maximum, when

$$c - 2z = 0, \text{ and } \therefore\ z = \frac{1}{2}c;$$

$$\therefore\ ax \text{ must be} > \frac{1}{4}c^2;$$

therefore the pump cannot work unless the length of the stroke be greater than $c^2 \div 4a$.

27. The same distance is divided in the three Thermometers into 180°, 100°, and 80° respectively. Hence indicating the respective Thermometers by F, C, R, we have

$$1^\circ(F) : 1^\circ(C) : 1^\circ(R) = \frac{1}{180} : \frac{1}{100} : \frac{1}{80}$$

$$= \frac{1}{9} : \frac{1}{5} : \frac{1}{4} = 20 : 36 : 45.$$

30. Let V be the volume of the bulb or part below the freezing point.

Then $180 \times {\cdot}0001\,V$ = expansion from freezing to boiling point

$$= 7\pi\left(\frac{1}{20}\right)^2;$$

$$\therefore\ V = \frac{7\pi}{400 \times {\cdot}018} = \frac{7\pi}{7{\cdot}2} = 3{\cdot}054 \text{ cub. in.}$$

HYDRODYNAMICS.

Efflux of Fluids from Vessels.

Ex. 9.

1. Let v be the velocity of the fluid at the aperture;

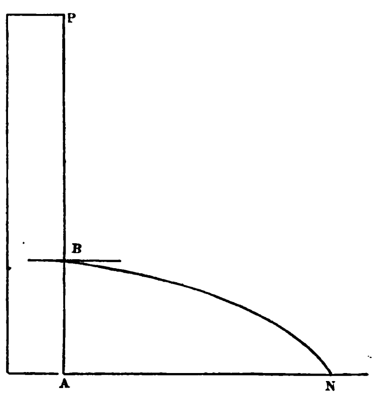
$$\text{area of aperture} = \frac{\pi}{4}(\cdot 125)^2 = \frac{\pi}{256},$$

$$\text{volume of cylinder} = \frac{\pi}{4} \times 5^2 \times 12 = 75\pi.$$

Hence $v \times 90 \times \frac{\pi}{256} = 75\pi$, since the time of filling $= 90$ sec.;

$$\therefore\ v = \frac{75 \times 256}{90} = 17\tfrac{7}{9} \text{ ft.}$$

3. Let $AP = x_1$, $AB = 3$, $AN = 5$.



The issuing fluid describes a parabola

$$y = x \tan\alpha - \frac{x^2}{4h\cos^2\alpha},$$

in which $\alpha = 0$, PB or $h = x_1 - 3$,

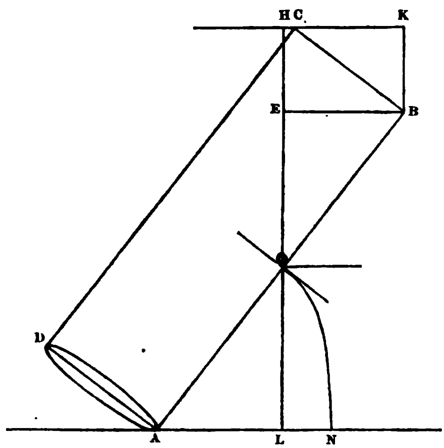
and when $y = -3$, $x = 5$;

$$\therefore\ -3 = -\frac{5^2}{4(x_1 - 3)};$$

$$\therefore\ x_1 = 3 + \frac{25}{12} = 5\tfrac{1}{12} \text{ ft.}$$

Similarly, x_2, x_3 may be found.

6. Let $ABCD$ be the cylinder, full of water, and inclined at an angle $BAN = 60°$ to the horizon,



$AB = 18$ ft., $AD = BC = 5$ ft., O the orifice at the middle of AB.

The initial velocity of the issuing stream is equal to that acquired by a particle falling down $HO = OE + BK$;

$$\therefore \text{ the impetus } h = 9 \sin 60° + 5 \sin 30°.$$

The equation to the parabolic curve described is

$$y = x \tan \alpha - \frac{x^2}{4h \cos^2 \alpha},$$

where $\alpha = -30°$, $y = OL = -9 \sin 60°$, $LN = x$;

$$\therefore -9\frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}x - \frac{x^2}{3\left(9\frac{\sqrt{3}}{2}+\frac{5}{2}\right)};$$

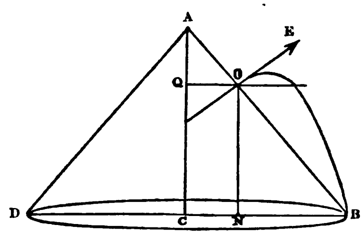
$$\therefore x^2 + \frac{1}{2}(27 + 5\sqrt{3})\,x = \frac{1}{4}(729 + 135\sqrt{3}),$$

whence x is found $= 8{\cdot}978$, and then

$$AN = AL + LN = 9 \cos 60° + x = 13{\cdot}478 \text{ ft.}$$

9. Let ABD be the cone, $\angle DAB = 90°$, $AC = 7$ ft.

O the orifice at a vertical depth $AQ = z$.



Now the pressure of the fluid at any point O on the concave surface of cone being in the direction of the normal at that point, on making an orifice the fluid will spout in the direction OE perpendicular to AB.

The equation to the curve of the issuing stream is

$$y = x\tan\alpha - \frac{x^2}{4h\cos^2\alpha}, \text{ in which } \alpha = 45°,\ h = z;$$

and in order that the curve may pass through B we must have $y = ON = -(7 - z)$, and $x = NB = NO = 7 - z$;

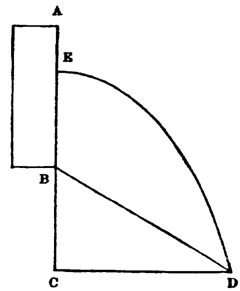
$$\therefore -(7-z) = 7 - z - \frac{(7-z)^2}{4z \times \frac{1}{2}};$$

$$\therefore 7 - z = 4z; \quad \therefore z = 1{\cdot}4;$$

$$\therefore AO = AQ \sec 45° = 1{\cdot}4 \times \sqrt{2}.$$

12. Let AB be the cylinder standing on the top of the plane BCD: and let E be the point at which the orifice must be made.

Let $AE = x$, $CD = y$;



then $EB = 10 - x$, $BC = y \times \dfrac{y}{\sqrt{3}}$.

Now $CD^2 = 4AE \times EC$,

$$\text{or } y^2 = 4x\left(10 - x + \frac{y}{\sqrt{3}}\right);$$

$$\therefore\ y^2 - \frac{4x}{\sqrt{3}}y = 4x(10 - x);$$

$$\therefore\ \left(y - \frac{2x}{\sqrt{3}}\right)^2 = \frac{4}{3}(30x - 2x^2);$$

$$\therefore\ y = \frac{2}{\sqrt{3}}\{x \pm (30x - 2x^2)^{\frac{1}{2}}\};$$

$$\therefore\ BD = \frac{4}{3}\{x \pm (30x - 2x^2)^{\frac{1}{2}}\} = \text{a maximum};$$

$$\therefore\ 1 \pm \frac{30 - 4x}{2(30x - 2x^2)^{\frac{1}{2}}} = 0,$$

$$\text{whence } 6x^2 - 90x = -225,$$

$$\text{which gives } x = 3{\cdot}17;$$

the less value alone being admissible.

15. Let k, k' denote the area of the orifice in the vessels; and x, y the altitudes; then K being the area of the base in each case, we have

time of emptying *first* vessel

$$= \sqrt{\frac{4x}{g}} \times \frac{K}{k},$$

and time of emptying second vessel

$$= \sqrt{\frac{4y}{g}} \times \frac{K}{k'}.$$

Now by the question the times are equal, and

$$k : k' = 2 : 1, \text{ or } k = 2k'.$$

Therefore, substituting, we get

$$\sqrt{x} = 2\sqrt{y};$$

$$\therefore\ x : y = 4 : 1.$$

19. Let x be the depth of the orifice; then the latus rectum of the parabola first described by the spouting fluid is $4x$; and the time of emptying a depth x is

$$= \frac{2K}{k} \cdot \sqrt{\frac{x}{2g}},$$

$$\text{but } K : k = 12 : 1, \text{ or } K = 12k;$$

$$\therefore\ \text{time} = 24\sqrt{\frac{x}{2g}};$$

therefore the time of one vibration of the pendulum

$$= 8\sqrt{\frac{x}{2g}};$$

therefore, if l be the length of the pendulum, we have

$$\pi\sqrt{\frac{l}{g}} = 8\sqrt{\frac{x}{2g}},$$

$$\text{whence } l = \frac{32x}{\pi^2},$$

hence, the ratio required

$$= 4x : \frac{32x}{\pi^2} = \pi^2 : 8.$$

22. Let k the orifice $= \pi a^2$, $AC = b$, $BE = 2R$,

$$AP = p,\ AQ = q,\ AB = h,$$

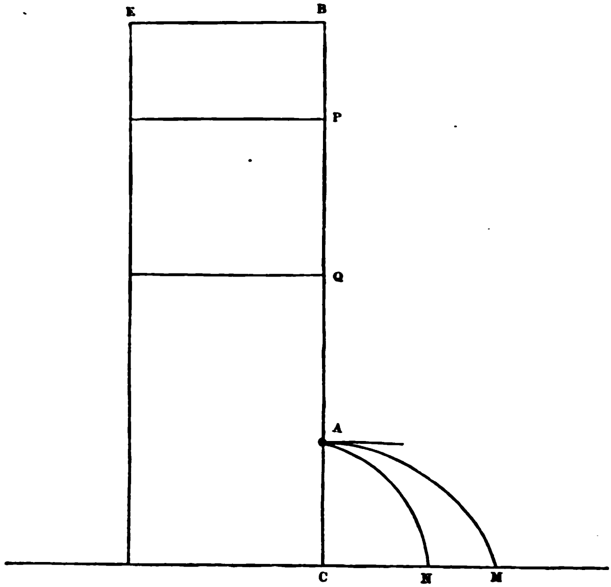
$$CM = m,\ CN = n.$$

$$\text{Then } y = x\tan\alpha - \frac{x^2}{4h\cos^2\alpha},$$

the equation to curve described, where $\alpha = 0$.

$$\text{Hence } -b = -\frac{m^2}{4p},$$

$$\text{and } -b = -\frac{n^2}{4q}.$$



$$\text{Now } r = \frac{K}{k}\left(\frac{\sqrt{2h} - \sqrt{2p}}{\sqrt{g}}\right),$$

$$\text{and } s = \frac{K}{k}\left(\frac{\sqrt{2h} - \sqrt{2q}}{\sqrt{g}}\right);$$

$$\therefore \frac{r}{s} = \frac{\sqrt{2h} - \dfrac{m}{\sqrt{2b}}}{\sqrt{2h} - \dfrac{n}{\sqrt{2b}}},$$

$$\text{since } \sqrt{2p} = \frac{m}{\sqrt{2b}}, \quad \sqrt{2q} = \frac{n}{\sqrt{2b}};$$

$$\therefore \frac{\sqrt{2h} - \dfrac{m}{\sqrt{2b}}}{\dfrac{m}{\sqrt{2b}} - \dfrac{n}{\sqrt{2b}}} = \frac{r}{s - r};$$

$$\therefore \sqrt{2h} = \frac{m}{\sqrt{2b}} + \frac{m - n}{\sqrt{2b}} \times \frac{r}{s - r} = \frac{1}{\sqrt{2b}}\left(\frac{ms - nr}{s - r}\right);$$

$$\therefore\ AB \text{ or } h = \frac{1}{4b}\left(\frac{ms - nr}{s - r}\right)^2.$$

$$\text{Again, } \frac{K}{k}\left(\frac{\sqrt{2p} - \sqrt{2q}}{\sqrt{g}}\right) = s - r;$$

$$\therefore\ R^2 = a^2 . \sqrt{g} . \frac{s - r}{\frac{m - n}{\sqrt{2b}}};$$

$$\therefore\ R = a\left(\frac{s - r}{m - n}\right)^{\frac{1}{2}}(2bg)^{\frac{1}{4}} = a\left\{2bg\left(\frac{s - r}{m - n}\right)^2\right\}^{\frac{1}{4}}.$$

25. The general formula is

$$t = \frac{1}{k\sqrt{2g}}\int \frac{-Kdx}{x^{\frac{1}{2}}},$$

and a cone being a solid of revolution, we have

$$K = \pi y^2;$$

also, from the geometry of the figure, we get

$$x : y - 5 :: 24 : 4 :: 6 : 1,$$

$$\text{whence } y^2 = \frac{(x + 30)^2}{36};$$

$$\therefore\ t = -\frac{\pi}{k} . \frac{1}{\sqrt{g}} . \frac{1}{36\sqrt{24}}\int_0^{24} \frac{(x + 30)^2}{x^{\frac{1}{2}}}\, dx,$$

$$\int_x \frac{(x + 30)^2}{x^{\frac{1}{2}}} = \int_x (x^{\frac{3}{2}} + 60x^{\frac{1}{2}} + 900x^{-\frac{1}{2}})$$

$$= \frac{2}{5}x^{\frac{5}{2}} + 40x^{\frac{3}{2}} + 1800x^{\frac{1}{2}};$$

therefore, between limits we have

$$t = \frac{\pi}{k} . \frac{1}{\sqrt{g}} . \frac{1}{\sqrt{24}}\left\{\frac{32}{5} + \frac{80}{3} + 50\right\}\sqrt{24}$$

$$= \frac{1246}{15k} . \frac{\pi}{\sqrt{g}}.$$

28.
$$t=-\frac{\pi}{k}\cdot\frac{1}{\sqrt{2g}}\int\frac{y^2dx}{x^{\frac{1}{2}}}$$

$$=-\frac{\pi}{k}\cdot\frac{1}{\sqrt{2g}}\int_0^r\frac{2rx-x^2}{x^{\frac{1}{2}}}dx$$

$$=-\frac{\pi}{k}\cdot\frac{1}{\sqrt{2g}}\int_0^r(2rx^{\frac{1}{2}}-x^{\frac{3}{2}})\,dx$$

$$=\frac{\pi}{k\sqrt{2g}}\left(\frac{4}{3}r^{\frac{5}{2}}-\frac{2}{5}r^{\frac{5}{2}}\right)$$

$$=\frac{\pi}{k\sqrt{2g}}\cdot\frac{14}{15}r^{\frac{5}{2}}$$

$$=\frac{7\pi r^2}{15k}\left(\frac{2r}{g}\right)^{\frac{1}{2}}.$$

31. Let $a=$ the height of the segment; also let t, t' be the times of emptying through the orifice in the vertex and base; then

$$t=-\frac{\pi}{k\sqrt{2g}}\int\frac{y^2dx}{x^{\frac{1}{2}}}=-\frac{\pi}{k\sqrt{2g}}\int\frac{2rx-x^2}{x^{\frac{1}{2}}}dx$$

$$=-\frac{\pi}{k\sqrt{2g}}\int(2rx^{\frac{1}{2}}-x^{\frac{3}{2}})\,dx\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

$$\text{also } t'=-\frac{\pi}{k\sqrt{2g}}\int\frac{2r(a-x)-(a-x)^2}{x^{\frac{1}{2}}}dx$$

$$=-\frac{\pi}{k\sqrt{2g}}\int\{a(2r-a)x^{-\frac{1}{2}}-2(r-a)x^{\frac{1}{2}}-x^{\frac{3}{2}}\}\,dx\ldots(2);$$

therefore by the question, we have

$$\int_0^a(2rx^{\frac{1}{2}}-x^{\frac{3}{2}})\,dx:\int_0^a\{a(2r-a)x^{-\frac{1}{2}}-2(r-a)x^{\frac{1}{2}}-x^{\frac{3}{2}}\}\,dx=2:3,$$

$$\text{or } \frac{2}{5}a^{\frac{5}{2}}-\frac{4}{3}ra^{\frac{3}{2}}:\frac{2}{5}a^{\frac{5}{2}}+\frac{4}{3}(r-a)^{\frac{3}{2}}-2a(2r-a)a^{\frac{1}{2}}=2:3,$$

whence, reducing, we get

$$3a-10r:8a-20r=2:3,$$

whence $9a - 30r = 16a - 40r$;

$$\therefore\ a = \frac{10}{7} r;$$

therefore volume of segment : volume of sphere

$$= \frac{\pi}{6}\left(6r - \frac{20r}{7}\right)\frac{100r^2}{49} : \frac{4\pi r^3}{3}$$

$$= \frac{22 \times 100r^3}{2 \times 7 \times 49} : 4r^3 = 275 : 343.$$

34. Let $y = Ax^n$ be assumed as the equation to the required parabola. Also let $a =$ the common altitude of the paraboloid and cylinder; then

$$t = \frac{\pi}{k(2g)^{\frac{1}{2}}}\int\frac{y^2 dx}{x^{\frac{1}{2}}}$$

$$= \frac{\pi A^2}{k\sqrt{2g}}\cdot\int\frac{x^{2n}}{x^{\frac{1}{2}}}dx = \frac{\pi A^2}{k\sqrt{2g}}\cdot\frac{x^{2n+\frac{1}{2}}}{2n + \frac{1}{2}}$$

between $x = 0$ and $x = a = \dfrac{2\pi A^2}{(4n+1)k\sqrt{2g}}\cdot a^{\frac{4n+1}{2}}$.

For the cylinder, we have

$$K = \pi A^2 . a^{2n},$$

$$\text{and } t' = \frac{2\pi A^2 a^{2n}}{k\sqrt{2g}} \times a^{\frac{1}{2}};$$

$\therefore\ t : t' = 1 : 4n + 1 = 1 : 9$ by the question,

hence $n = 2$,

and the equation to the parabola becomes

$$y = Ax^2.$$

36. Let the equation to the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the axis of z being vertical. Then, for the velocity of the descending surface, we have

$$\frac{dz}{dt} = \frac{k\{2g(c-z)\}^{\frac{1}{2}}}{K};$$

putting the equation to the ellipsoid under the form

$$\frac{x^2}{a^2\left(1-\frac{z^2}{c^2}\right)} + \frac{y^2}{b^2\left(1-\frac{z^2}{c^2}\right)} = 1,$$

$$\text{we have } K = \pi ab\left(1-\frac{z^2}{c^2}\right);$$

$$\therefore\ dt = \frac{\pi ab}{k(2g)^{\frac{1}{2}}} \cdot \frac{1-\frac{z^2}{c^2}}{(c-z)^{\frac{1}{2}}}\,dz$$

$$= \frac{\pi ab}{kc^2(2g)^{\frac{1}{2}}}(c+z)(c-z)^{\frac{1}{2}}\,dz.$$

$$\text{Hence } t = \frac{\pi ab}{kc^2(2g)^{\frac{1}{2}}}\int_{-c}^{c}\{2c-(c-z)\}(c-z)^{\frac{1}{2}}\,dz$$

$$= \frac{\pi ab}{kc^2(2g)^{\frac{1}{2}}}\int_{-c}^{+c}\left\{-\frac{4}{3}c(c-z)^{\frac{3}{2}} + \frac{2}{5}(c-z)^{\frac{5}{2}}\right\}$$

$$= \frac{\pi ab}{kc^2(2g)^{\frac{1}{2}}}\left(\frac{4}{3}c \times 2^{\frac{3}{2}} \times c^{\frac{3}{2}} - \frac{2}{5} \times 2^{\frac{5}{2}} \times c^{\frac{5}{2}}\right)$$

$$= \frac{\pi abc^{\frac{1}{2}}}{k(2g)^{\frac{1}{2}}}\left(\frac{1}{3} \times 2^{\frac{7}{2}} - \frac{1}{5} \times 2^{\frac{7}{2}}\right)$$

$$= \frac{16\pi ab}{15k}\left(\frac{c}{g}\right)^{\frac{1}{2}}.$$

37. The solid being one of revolution, we have

$$t = -\frac{\pi}{k\sqrt{2g}}\int\frac{y^2dx}{x^{\frac{1}{2}}}$$

$$= -\frac{\pi}{k(2g)^{\frac{1}{2}}}\int y^2x^{-\frac{1}{2}}\,dx = -\frac{\pi}{k(2g)^{\frac{1}{2}}}\left(2y^2x^{\frac{1}{2}} - 4\int yx^{\frac{1}{2}}\,dy\right),$$

$$\int yx^{\frac{1}{2}}\,dy = \int y\,(2a-x)\,dx = -\frac{2}{3}y\,(2a-x)^{\frac{3}{2}} + \frac{2}{3}\int (2a-x)^{\frac{3}{2}}\,dy,$$

$$\int (2a-x)^{\frac{3}{2}}\,dy = \int \frac{(2a-x)^2}{x^{\frac{1}{2}}}\,dx = 8a^2x^{\frac{1}{2}} - \frac{8}{3}ax^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + C;$$

$$\therefore\ t = -\frac{\pi}{k\,(2g)^{\frac{1}{2}}}\left\{2y^2x^{\frac{1}{2}} + \frac{8}{3}y\,(2a-x)^{\frac{3}{2}} - \frac{8}{3}\left(8a^2x^{\frac{1}{2}} - \frac{8}{3}ax^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}\right)\right\},$$

$$\text{and from } \begin{cases} x = 0 \text{ to } x = 2a, \\ y = 0 \text{ to } y = \pi a, \end{cases}$$

we obtain, for the whole efflux,

$$t = \frac{\pi a^{\frac{5}{2}}}{k\,(g)^{\frac{1}{2}}}\left\{2\pi^2 - \frac{8^3}{45}\right\} = \frac{\pi a^2}{k}\left(\frac{a}{g}\right)^{\frac{1}{2}}\left(2\pi^2 - \frac{8^3}{45}\right).$$

38. Let πy^2 be any circular section of the vessel whose altitude is x; and k the area of the orifice; then the velocity at the orifice is

$$\sqrt{gx},$$

and that of the descending surface is

$$= \frac{k\sqrt{g}}{\pi y^2}\sqrt{x} = \text{constant quantity};$$

$$\therefore\ y^2 = \sqrt{x} \times \text{constant} = \sqrt{Ax} \text{ suppose};$$

$$\therefore\ y^4 = Ax,$$

the equation to a parabola of the fourth order.

40. The velocity of efflux, being that due to an altitude of 27690 feet, will

$$= (2g \times 27690)^{\frac{1}{2}}$$
$$= 1335{\cdot}17 \text{ ft. nearly.}$$

42. Let ρ = the density of the external air, and therefore the density of the air in the receiver at first; and suppose E to be the elastic force of the external air; then the density of the air in the receiver after 15 descents of the piston is

$$\left(\frac{10}{11\frac{1}{2}}\right)^{15}\rho = \left(\frac{20}{23}\right)^{15}\rho,$$

and its corresponding elastic force is

$$\left(\frac{20}{23}\right)^{15} \times E.$$

Now if v be the initial velocity, we have

$$v = \left\{2g\frac{\rho - \left(\frac{20}{23}\right)^{15}\rho}{\rho} \times 27690\right\}^{\frac{1}{2}}$$

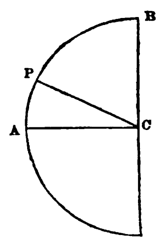
$$= \left[\left\{1 - \left(\frac{20}{23}\right)^{15}\right\} \times 64{\cdot}3817 \times 27690\right]^{\frac{1}{2}}$$

$$= \{{\cdot}877105 \times 64{\cdot}3817 \times 27690\}^{\frac{1}{2}}$$

$$= 1250{\cdot}45 \text{ ft. nearly.}$$

RESISTANCES.

Ex. 10.



1. Let C be the middle point of the rectilinear boundary of either face of the lamina, P any point in the curvilinear circumference of this face, and CA its axis.

Let $AC = a$, $\angle ACP = \theta$, arc $AP = s$, $\tau =$ the thickness of the lamina. Let P, Q denote the resultant pressures of the fluid on the lamina, accordingly as A or C meets the stream.

Then, $v \cos\theta$ being the normal component of the velocity (v) with which the fluid impinges upon the lamina at P, the component of the pressure upon an elemental rectangle τds of the edge of the lamina, parallel to AC, will be equal to

$$\frac{1}{2}\rho . \tau ds . (v\cos\theta)^2 . \cos\theta = \frac{1}{2}\rho\tau a v^2 \cos^3\theta\, d\theta.$$

Hence, P being equal to twice the component of the pressure upon the quadrant AB, parallel to AC, we have

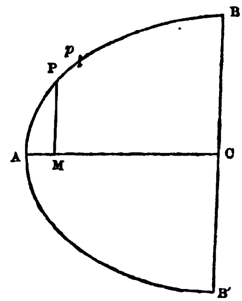
$$P = \rho\tau a v^2 \int_0^{\frac{1}{2}\pi} \cos^3\theta\, d\theta = \rho\tau a v^2 \int_0^1 (1 - \sin^2\theta)\, d\sin\theta$$

$$= \rho\tau a v^2\left(1 - \frac{1}{3}\right) = \frac{2}{3}\rho\tau a v^2.$$

The pressure Q upon the base of the lamina will be equal to

$$\frac{1}{2}\rho . 2\tau a . v^2 = \rho\tau a v^2.$$

Hence $P : Q = 2 : 3$.



3. Let a be the length of the semi-axis major AC, $2b$ the length of the axis minor BB', of either face of the lamina. Let CA, CB, produced be taken as co-ordinate axes of x and y respectively; let $CM = x$, $PM = y$, $AP = s$, $Pp = ds$, p being any point of the curve AB indefinitely near to P. Then, τ denoting the thickness of the lamina, and ρ the density of the fluid, the normal resistance on the portion Pp of the lamina, the vertex A being supposed to move foremost, will be equal to

$$\frac{1}{2}\rho\tau v^2 \frac{dy^2}{ds^2} ds,$$

and, the component of this, parallel to AC, will be equal to

$$\frac{1}{2}\rho\tau v^2 \frac{dy^3}{ds^3} ds.$$

Hence, if P be the resultant resistance on the lamina parallel to AC, we have

$$P = \frac{1}{2}\rho\tau v^2 \int \frac{dy^3}{ds^3} ds = \frac{1}{2}\rho\tau v^2 \int \frac{dy}{1 + \dfrac{dx^2}{dy^2}}.$$

But from the equation to the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

we may get $\dfrac{dx^2}{dy^2} = \dfrac{a^2}{b^2} \cdot \dfrac{y^2}{b^2 - y^2}$;

hence, integrating between the limits $-y$ and $+y$, we have

$$P = \frac{1}{2}\rho\tau v^2 b^2 \int \frac{(b^2 - y^2)\, dy}{b^4 + (a^2 - b^2)\, y^2}$$

$$= \frac{1}{2}\rho\tau v^2 \frac{b^2}{a^2 - b^2} \int \left\{\frac{a^2 b^2}{b^4 + (a^2 - b^2)\, y^2} - 1\right\} dy$$

$$= \frac{1}{2}\rho\tau v^2 \frac{2b^2}{a^2 - b^2} \left\{\frac{a^2}{(a^2 - b^2)^{\frac{1}{2}}} \tan^{-1} \frac{(a^2 - b^2)^{\frac{1}{2}} y}{b^2} - y\right\},$$

or, b being the value of y in the limit,

$$P=\frac{1}{2}\rho\tau v^2\frac{2b^2}{a^2-b^2}\left\{\frac{a^2}{(a^2-b^2)^{\frac{1}{2}}}\tan^{-1}\frac{(a^2-b^2)^{\frac{1}{2}}}{b}-b\right\}.$$

But, if Q be the resistance when the motion of the lamina takes place with its base foremost,

$$Q=\frac{1}{2}\rho\tau v^2 . 2b;$$

$$\therefore \frac{P}{Q}=\frac{a^2b}{(a^2-b^2)^{\frac{3}{2}}}\tan^{-1}\left(\frac{a^2}{b^2}-1\right)^{\frac{1}{2}}-\frac{b^2}{a^2-b^2}.$$

6. Let r be the radius of the sphere; the centre being the origin of co-ordinates; then the resistance

$$=\pi\rho v^2\int y dy\frac{dy^2}{ds^2}.$$

Now $y^2=r^2-x^2$, and $\frac{ds}{dy}=\left(1+\frac{dx^2}{dy^2}\right)^{\frac{1}{2}}$

$$=\left(1+\frac{y^2}{x^2}\right)^{\frac{1}{2}}=\frac{r}{x};$$

$$\therefore \int y dy\frac{dy^2}{ds^2}=\int y dy\frac{x^2}{r^2}$$

$$=\int y\left(1-\frac{y^2}{r^2}\right)dy$$

$$=\left(\frac{1}{2}-\frac{1}{4}.\frac{y^2}{r^2}\right)y^2+C,$$

which, between the limits $y=0$, $y=r$,

$$=\frac{1}{4}r^2;$$

$$\therefore \text{ resistance on the sphere }=\frac{1}{4}\pi\rho v^2r^2,$$

and the resistance on the circular plate is

$$=\frac{1}{2}\pi\rho v^2r^2;$$

therefore resistance on sphere : resistance on plate = 1 : 2.

8. The sections of a cylinder made by planes parallel to the base are circles. Hence if R denote the resistance upon the diameter (d) of one of these sections, and l the length of the cylinder; the resistance upon the cylinder when moving in a direction perpendicular to its axis is

$$=\frac{2}{3}R\times l,$$

also the resistance upon the base of the cylinder when it moves in the direction of its axis is

$$=lR\times\frac{\pi\frac{d^2}{4}}{ld}=\pi R\frac{d}{4}.$$

On comparing the latter resistance with the former, we obtain the ratio

$$\frac{\pi d}{4}\times R\div\frac{2l}{3}\times R=\frac{3\pi d}{8l}.$$

11. Let $2a$, $2b$ be the axes of the generating ellipse; v the velocity of the spheroid, and ρ the density of the fluid; then

$$a-x=\frac{a}{b}(b^2-y^2)^{\frac{1}{2}};$$

$$\therefore\ -\frac{dx}{dy}=\frac{a}{b}\cdot\frac{-y}{(b^2-y^2)^{\frac{1}{2}}};$$

$$\therefore\ 1+\frac{dx^2}{dy^2}=\frac{b^4-b^2y^2+a^2y^2}{b^2(b^2-y^2)}=\frac{b^4+a^2e^2y^2}{b^2(b^2-y^2)};$$

$$\therefore\ \text{resistance}=\pi\rho v^2b^2\int\frac{(b^2-y^2)\,ydy}{b^4+a^2e^2y^2}$$

$$=\frac{\pi\rho v^2b^2}{a^2e^2}\int\frac{(b^2a^2e^2-a^2e^2y^2+b^4-b^4)\,ydy}{b^4+a^2e^2y^2}$$

$$=\frac{\pi\rho v^2b^2}{a^2e^2}\left\{\int\frac{(b^2a^2e^2+b^4)\,ydy}{b^4+a^2e^2y^2}-\int ydy\right\},$$

and between the limits $y=0$, $y=b$,

$$\text{resistance}=\frac{\pi\rho v^2b^4}{2a^2e^2}\left\{\left(\frac{b^2}{a^2e^2}+1\right)\log\left(1+\frac{a^2e^2}{b^2}\right)-1\right\}$$

$$=\frac{\pi\rho v^2b^4}{2(a^2-b^2)}\left(\frac{2a^2}{a^2-b^2}\log\frac{a}{b}-1\right).$$

12. By the nature of the cycloid, we have

$$x = a(1 - \cos\theta), \quad y = a(\theta + \sin\theta);$$

$$\therefore \frac{dx}{dy} = \frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}; \quad \therefore 1 + \left(\frac{dx}{dy}\right)^2 = \sec^2\frac{\theta}{2}.$$

Also $ydy = a^2(\theta + \sin\theta)(1 + \cos\theta)\,d\theta = 2a^2(\theta + \sin\theta)\cos^2\frac{\theta}{2}\,d\theta$;

$$\therefore \text{resistance} = 2\pi\rho v^2 a^2 \int (\theta + \sin\theta)\cos^4\frac{\theta}{2}\,d\theta.$$

$$\text{Now } \cos^4\frac{\theta}{2} = \frac{1}{8}\{\cos 2\theta + 4\cos\theta + 3\};$$

$$\therefore \int \cos^4\frac{\theta}{2}\,\theta\,d\theta = \frac{1}{8}\left\{\frac{1}{4}(2\theta\sin 2\theta + \cos 2\theta) + 4(\theta\sin\theta + \cos\theta) + \frac{3}{2}\theta^2\right\}.$$

$$\text{Also } \int \sin\theta \,.\, \cos^4\frac{\theta}{2}\,d\theta = -4\int \cos^5\frac{\theta}{2}\,d\cos\frac{\theta}{2} = -\frac{2}{3}\cos^6\frac{\theta}{2};$$

$\therefore$ resistance

$$= 2\pi\rho v^2 a^2\left(\frac{\theta\sin 2\theta}{16} + \frac{\cos 2\theta}{32} + \frac{\theta\sin\theta}{2} + \frac{\cos\theta}{2} + \frac{3\theta^2}{16} - \frac{2}{3}\cos^6\frac{\theta}{2}\right) + C,$$

and taking the integral from $\theta = 0$, to $\theta = \pi$,

$$\text{resistance} = 2\pi\rho v^2 a^2\left\{\frac{3\pi^2}{16} - \frac{1}{3}\right\},$$

and when the solid moves base foremost,

$$\text{resistance} = \frac{1}{2}\rho v^2 \pi \times \pi^2 a^2;$$

therefore the resistances are as

$$2\pi\rho v^2 a^2\left(\frac{9\pi^2 - 16}{48}\right) : \frac{1}{2}\rho v^2 \pi^3 a^2 = 9\pi^2 - 16 : 12\pi^2.$$

APPENDIX.

Ex. 16 [20], p. 79. Companion, p. 91.

Here the functions are,

$$
\begin{aligned}
X &= x^6 + 24x^5 + 125x^4 - 376x^3 - 1726x^2 + 5592x - 4080,\\
X_1 &= 3x^5 + 60x^4 + 250x^3 - 564x^2 - 1726x + 2796,\\
X_2 &= 5x^4 + 68x^3 + 52x^2 - 908x + 1018{\cdot}4,\\
X_3 &= 529x^3 + 2736x^2 - 14370{\cdot}74x + 13934{\cdot}87,\\
X_4 &= 84285 75x^2 - 29399158{\cdot}695x + 25636304{\cdot}35,\\
X_5 &= 3295138317 0970x - 57492539569293,\\
X_6 &= +,
\end{aligned}
$$

which shew that *all the roots are real,* three positive and three negative. The limits are 2 and -14.

	X	X_1	X_2	X_3	X_4	X_5	X_6
If $x=0$,	$-$	$+$	$+$	$+$	$+$	$-$	$+$
$x=1$,	$-$	$+$	$+$	$+$	$+$	$-$	$+$
$x=2$,	$+$	$+$	$+$	$+$	$+$	$+$	$+$

	X	X_1	X_2	X_3	X_4	X_5	X_6	
If $x=0$,	$-$	$+$	$+$	$+$	$+$	$-$	$+$	three changes,
$x=-1$,	$-$	$+$	$+$	$+$	$+$	$-$	$+$	....
$x=-2$,	$-$	$+$	$+$	$+$	$+$	$-$	$+$	....
$x=-3$,	$-$	$+$	$+$	$+$	$+$	$-$	$+$	....
$x=-4$,	$-$	$-$	$+$	$+$	$+$	$-$	$+$	....
$x=-5$,	$-$	$-$	$+$	$+$	$+$	$-$	$+$	....
$x=-6$,	$+$	$-$	$+$	$+$	$+$	$-$	$+$	four changes,
$x=-7$,	$+$	$-$	$-$	$+$	$+$	$-$	$+$	....
$x=-8$,	$+$	$-$	$-$	$+$	$+$	$-$	$+$	....
$x=-9$,	$+$	$+$	$-$	$-$	$+$	$-$	$+$	....
$x=-10$,	$-$	$+$	$-$	$-$	$+$	$-$	$+$	five changes,
$x=-11$,	$-$	$+$	$-$	$-$	$+$	$-$	$+$	....
$x=-12$,	$-$	$+$	$+$	$-$	$+$	$-$	$+$	....
$x=-13$,	$-$	$-$	$+$	$-$	$+$	$-$	$+$	....
$x=-14$,	$+$	$-$	$+$	$-$	$+$	$-$	$+$	six changes;

$\therefore$ the positions of the roots are expressed by

$$\{-14, -13\};\ \{-10, -9\};\ \{-6, -5\};\ \{1, 2\};\ \{1, 2\};\ \{1, 2\}.$$

In fact, the roots, worked out by Horner's Method, are

$$-13{\cdot}74596\ldots\ldots;\quad -9{\cdot}74456\ldots\ldots;\quad -5{\cdot}74165\ldots\ldots;\quad 1{\cdot}74165;$$
$$1{\cdot}74456\ldots\ldots;\quad 1{\cdot}74596\ldots\ldots.$$

THE END.

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